Mean-field approximations for short-range four-body interactions at $\nu = \frac{3}{5}$

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Trial wave functions such as the Moore-Read and Read-Rezayi states, which minimize short-range multibody interactions, are candidate states for describing the fractional quantum Hall effects at filling factors $\nu = 1/2$ and $2/5$ in the second Landau level. These trial wave functions are unique zero-energy states of three-body and four-body interaction Hamiltonians, respectively, but they are not close to the ground states of the Coulomb interaction. Previous studies using extensive parameter scans have found optimal two-body interactions on the sphere that produce states close to these. Here we focus on short-ranged four-body interaction and study two mean-field approximations that reduce the four-body interactions to two-body interactions on the sphere by replacing composite operators with their incompressible ground-state expectation values. We present the results for pseudopotentials of these approximate interactions. A comparison of finite system spectra on the sphere of the four-body and the approximate interactions at filling fraction $\nu = 3/5$ shows that these approximations produce good effective descriptions of the low-energy structure of the four-body interaction Hamiltonian. The approach also independently reproduces the optimal two-body interaction inferred from parameter scans. We also show that for $n = 3$, but not for $n = 4$, the mean-field approximations of the n-body interaction are equivalent to particle-hole symmetrization of the interaction. Within the system sizes accessible, analysis of the spectrum of the mean-field two-body Hamiltonian on the torus was inconclusive, and indicates a competing anisotropic state in the system.

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I. INTRODUCTION

The physics of electrons confined to two dimensions in the limit of high magnetic fields is described by a Hamiltonian that contains no kinetic energy but only the Coulomb interaction term, with the kinetic energy indirectly manifesting itself through the holomorphic nature of the Hilbert space. Interacting electrons in this Hilbert space exhibit a rich set of topological and conventionally ordered phases [1–3]. The phases in the lowest Landau level can be explained accurately using composite fermion wave functions [4,5]. The structure of the fractional quantum Hall effect in the second Landau level has been harder to explain using variational studies. Among the several candidate wave functions proposed to describe these states are a set of clustered states including the Pfaffian and the $k = 3$ Read-Rezayi states occurring at filling fractions $1/2$ and $3/5$ in this Landau level [6,7]. The correlations contained in these states are such that they minimize certain model Hamiltonians [7,8] that penalize specific short-range configurations of clusters of a few particles. These wave functions do not have large overlaps with the physical two-body Coulomb interaction ground states. The Moore-Read state, for instance, has an overlap of 0.69 with the Coulomb ground state in a system of size $N = 14$ [9]. However, it has been argued that both states capture the topological properties of the Coulomb ground states.

Since these states minimize a model interaction energy rather than the physical two-body Coulomb interaction energy, it is interesting to ask whether there is a two-body interaction that produces ground states that are close to these clustered states. One approach to addressing this is to consider general short-range two-body interactions parametrized by Haldane pseudopotentials and scan the parameter space to identify the optimal pseudopotentials that produce a homogeneous ground state with maximal overlap with the clustered state [10,11]. Another approach suggested in Ref. [12] is to make use of the model n-body interactions that annihilate these clustered states to arrive at approximate two-body interactions via a mean-field mapping. These two approaches surprisingly produce the same optimal interaction in the case of the Moore-Read state, which is annihilated by the three-body interaction [10].

In this study, we explore an extension of the mean-field approximation to the case of the four-body interaction that produces the $k = 3$ Read-Rezayi state as the ground state. There are two possible ways to map the four-body interaction to a two-body interaction: (i) by replacing two pairs of composite operators $c_i^\dagger c_j$, or (ii) by replacing $c_i^\dagger c_j^\dagger c_k c_l$ with their ground-state expectation values. The former method is scalable to larger system sizes, allowing us to extrapolate to the mean-field two-body pseudopotentials in the thermodynamic limit. Interestingly, the mean-field interaction matches exactly with what was obtained through an extensive parameter scan in Ref. [10]. Note that, in addition to the interactions whose influence we explore in this study, the state describing the physical system can be qualitatively changed by the presence of disorder, especially at filling fraction $\nu = \frac{1}{2}$ [13–16]. However, we consider spin-polarized systems that are disorder-free for tractability using finite system studies. We also note that even though a purely two-body mean-field interaction at filling fraction $\nu = 1/2$ may show a Pfaffian ground state at the right shift on the sphere, this may not stabilize a gapped phase in the thermodynamic limit without explicit particle-hole symmetry-breaking terms [17–19]. Such terms are not needed for states away from half-filling that we consider in this study.

The general $n > 2$-body interaction, and in particular the three-body interaction, is not particle-hole symmetric. Particle-hole symmetrization of the short-range three-body interaction Hamiltonian produces a two-body interaction that has a low-energy spectrum close to that of the original three-body interaction [18,20]. It was found in Ref. [12] that the mean-field two-body approximation reproduced the same two-body interaction as the symmetrization. We explain why the two interactions exactly reproduce the same spectra, and we extend this analysis to the case of the four-body interaction and show that this exact relation between symmetrization and the mean-field approximation is restricted to the case of the three-body interaction.

In Sec. II, we introduce the notion of Haldane pseudopotentials for general n-body interactions, followed by a description of the mean-field approximation, discussing the idea for the case of three- and four-body interactions. The mean-field approximation of the four-body interaction can be defined to produce a three-body or a two-body interaction. The latter can be arrived at in two ways. Every method results in a rotationally symmetric interaction and therefore can be specified in terms of the pseudopotentials. In Sec. III, we present the results of the pseudopotentials of the mean-field two- and three-body interactions for finite systems as well as in the large system limit. In Sec. IV, we discuss the relation between the mean-field approximation and the particle-hole symmetrization/antisymmetrization of the interactions. We show that the mean-field approximation to the three-body interaction and symmetrization produce the same spectrum. We show that this result does not generalize to the case of the four-body interaction. In Sec. V, we compare the finite system spectra of the mean-field interactions with that of the exact four-body interaction. The results suggest that the mean-field interaction closely reproduces the effective physics of the incompressible and few quasiparticle/quasihole systems in the spherical geometry. Results for the torus geometry are presented indicating a possibly competing anisotropic phase in the system. Approximate formulas for the mean-field interaction pseudopotentials that could be used in further numerical studies are presented in the Appendix A.

II. THE TYPES OF MEAN FIELD IN THE SPHERICAL GEOMETRY

We use the standard Haldane spherical geometry [21,22], in which N electrons are confined to the surface of a sphere of radius R, with a uniform, perpendicular magnetic field B being provided by a Dirac magnetic monopole of strength $2Q\phi_0$ ($2Q$ is an integer) placed at the center of the sphere, where the flux quanta ϕ_0 is hc/e. The corresponding magnetic length has a value $\ell_B = R/\sqrt{Q}$. The N-electron Hilbert space is spanned by the configurations $|\mathbf{p}\rangle = |p_1, p_2, \dots, p_N\rangle$ of electrons occupying orbitals p_i (with $p_i \in \{-Q, -Q+1, \dots, Q\}$). The general four-body Hamiltonian can be written as

$$\mathcal{H}^{(4)} = \sum_{p_i;q_i} V^{(4)}_{\mathbf{q};\mathbf{k}} c^\dagger_{p_4} c^\dagger_{p_3} c^\dagger_{p_2} c^\dagger_{p_1} c_{q_1} c_{q_2} c_{q_3} c_{q_4}, \tag{1}$$

where $\mathbf{p} = (p_1, p_2, p_3, p_4)$, $\mathbf{q} = (q_1, q_2, q_3, q_4)$, and the indices correspond to an ordered set of L_z quantum numbers of electrons ($p_i < p_{i+1}$ and $q_i < q_{i+1}$).

When the considered system has additional symmetries, certain constraints can be imposed reducing the number of independent parameters that describe the interaction. A rotationally symmetric four-body interaction of spinless fermions on the Haldane sphere can be described by a sequence of generalized Haldane four-body pseudopotentials $\{V_l^{(4)}\}_l$, where V_l is the energy needed for four particles to be in a relative angular momentum of $4Q - l$. Thus the Hamiltonian has the form

$$\mathcal{H}^{(4)} = \sum_{l=6}^{2Q} V_l^{(4)} P^{(4)}_{4Q-l}, \tag{2}$$

where $P^{(4)}_{4Q-l}$ is a projector onto the relative angular momentum $4Q - l$ subspace of four particles. This can be explicitly written as follows:

$$P^{(4)}_L = \sum_a \sum_{Lz=-L}^{L} \sum_{\mathbf{p},\mathbf{q}} \psi_{L,Lz,a}(\mathbf{p}) \psi_{L,Lz}(\mathbf{q}) \prod_{i=1}^{4} c^\dagger_{p_i} \prod_{i=1}^{4} c_{q_i}, \tag{3}$$

where $\psi_{L,Lz,a}(p_1, p_2, p_3, p_4)$ are the Clebsch-Gordan coefficients when the four-particle state of total angular momentum L and z-component angular momentum L_z is expanded in the single-particle basis $|p_1, p_2, p_3, p_4\rangle$, i.e.,

$$|\psi_{L,Lz,a}\rangle = \sum_{\mathbf{p}} \psi_{L,Lz,a}(p_1, p_2, p_3, p_4) |p_1, p_2, p_3, p_4\rangle. \tag{4}$$

Pauli exclusion implies that the allowed values of $l = 4Q - L$ for spinless fermions are $l = 6, 8, 9, \dots$. The index a corresponds to the possibility of different multiplets of angular momentum L. Short-range interaction corresponds to smaller values of l. Several independent angular momentum multiplets can occur for $l > 9$, and the additional quantum number a is then required [8]. However, in this article, for simplicity, we shall consider only those interactions for which $l = 6$ or 8, and the quantum number a will not be required. The wave functions $\psi^{L,Lz}_{\mathbf{p}}$ can be obtained via exact diagonalization of four particles in a generic rotationally symmetric two-body interaction.

In Ref. [12], the authors introduced a mean-field mapping of the three-body interaction to a two-body interaction by replacing a single quadratic composite operator $c^\dagger_{q_1} c_{k_1}$ with its expectation value. Analysis was restricted to states in the vicinity of the incompressible ground states, for which the expectation values take a simple form $\langle c^\dagger_{q_1} c_{k_1}\rangle = \nu\, \delta_{q_1,k_1}$ because of angular momentum conservation. We can apply this method here, resulting in a reduction of the four-body interaction to a three-body interaction.

The general four-body Hamiltonian [Eq. (1)] can be written without the restriction on the ordering of the single-particle

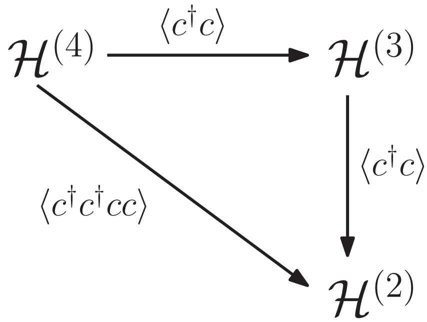

FIG. 1. A schematic diagram of two possible methods of MF reduction of four-body interaction to two-body interaction. The $\mathrm{MF}^2_{\langle c^\dagger c\rangle}$ method replaces a pair of creation and annihilation operators with their expectation value twice. As an intermediate step, one obtains the three-body operator $\mathcal{H}^{(3)}$. The diagonal arrow corresponds to the mean-field mapping using numerically obtained correlations from Read-Rezayi state $c^\dagger_{q_2}c^\dagger_{q_1}c_{k_1}c_{k_2}$.

angular momenta as

$$\mathcal{H}^{(4)} = \sum_{p_i,q_i=-Q}^{Q} c^\dagger_{p_4}c^\dagger_{p_3}c^\dagger_{p_2}c^\dagger_{p_1}\frac{V^{(4)}_{\mathbf{p},\mathbf{q}}}{4!4!}c_{q_1}c_{q_2}c_{q_3}c_{q_4}.$$

Antisymmetry of $V_{\mathbf{p},\mathbf{q}}$ is assumed under interchange of single-particle indices within $\mathbf{p}$ and $\mathbf{q}$. Upon applying the mean-field approximation, we obtain a three-body interaction of the form

$$\mathcal{H}^{(3)} = \sum_{p_i,q_i=-Q}^{Q} c^\dagger_{p_3}c^\dagger_{p_2}c^\dagger_{p_1}\frac{V^{(3)}_{p_1,p_2,p_3;q_1,q_2,q_3}}{3!3!}c_{q_1}c_{q_2}c_{q_3}, \quad (5)$$

where $V^{(3)}$ is given by the partial trace over one pair of indices,

$$V^{(3)}_{p_1,p_2,p_3;q_1,q_2,q_3} = \nu \sum_{p_4,q_4=-Q}^{Q} \delta_{p_4q_4}V^{(4)}_{\mathbf{p},\mathbf{q}}. \quad (6)$$

The three-body pseudopotentials $V^{(3)}_l$ of the mean-field three-body Hamiltonian can be obtained by numerically diagonalizing a system of three particles. The energy of the three-particle cluster of angular momentum $4Q-l$ gives the pseudopotential $V^{(3)}_l$.

A mean-field approximation of a similar kind applied now to the above three-body Hamiltonian results in a two-body interaction. The two-body pseudopotentials $V^{(2)}_l$ can now be obtained by diagonalizing a two-particle system. Thus reduction (four- to two-body) is obtained by applying a "single" mean-field approximation twice (see Fig. 1). Since we approximated operators $c^\dagger c$ with an expected value $\nu\delta_{q_1,k_1}$, we will denote this type of mean-field reduction by $\mathrm{MF}^2_{\langle c^\dagger c\rangle}$. The intermediate step of reduction of a four-body Hamiltonian to a three-body one by applying the approximation once is denoted $\mathrm{MF}_{\langle c^\dagger c\rangle}$.

One can construct an alternative mean-field reduction of four- to two-body Hamiltonians by replacing the composite quartic operator $c^\dagger_{q_2}c^\dagger_{q_1}c_{k_1}c_{k_2}$ with its expectation value in the incompressible ground state. Such an expectation value is not easy to calculate, even when one considers the homogeneous ground state. So we approach the problem with numerical calculations of correlations in the ground state. For each pair of indexes (q_1, q_2) we calculate the expected value of $c^\dagger_{q_2}c^\dagger_{q_1}c_{k_1}c_{k_2}$, which is later used to infer a mean-field two-body Hamiltonian. We will denote this mean-field mapping by $\mathrm{MF}_{\langle c^\dagger c^\dagger cc\rangle}$. In this study, we will use the 3/5 filling fraction to explore the mean field approximation, as an incompressible state ($k=3$ Read-Rezayi state) is produced by the short-range four-body interaction at this filling fraction [7].

The mean-field Hamiltonian needs to be rotationally symmetric in order to be able to calculate the pseudopotentials. It can be easily seen that the methods produce rotationally symmetric approximations. Due to the rotational symmetry of the original four-body interaction, the interaction parameters $V_{\mathbf{pq}}$ are elements of a linear combination of projections onto angular momentum subspaces. Therefore, these interaction parameters form a rotationally invariant tensor. The mean-field approximations $\mathrm{MF}_{\langle c^\dagger c\rangle}$ and $\mathrm{MF}^2_{\langle c^\dagger c\rangle}$ correspond to contraction of indices of this tensor with the indices of the rotationally invariant tensors $\delta_{p_4q_4}$ [Eq. (6)] and $\delta_{p_4q_4}\delta_{p_3q_3}$, respectively. Therefore, the mean-field interaction parameters $V^{(3)}_{p_1p_2p_3;q_1q_2q_3}$ and $V^{(2)}_{p_1p_2;q_1q_2}$ obtained in this way are rotationally invariant. Rotational invariance implies that the interaction parameters of the mean-field Hamiltonian are linear combinations of angular momentum projection operators, the coefficients of which give the pseudopotentials. In the case of $\mathrm{MF}_{\langle c^\dagger c^\dagger cc\rangle}$, $V_{\mathbf{pq}}$ is contracted with the correlation $\langle c^\dagger_{p_1}c^\dagger_{p_2}c_{q_1}c_{q_2}\rangle$, which is again rotationally symmetric due to the rotational symmetry ($L=0$) of the ground state. The information contained in the correlation function can indeed be represented as linear combinations of two-particle angular momentum projection operators (such expansions for specific finite systems are presented in Appendix B).

III. PSEUDOPOTENTIALS FOR THE MEAN-FIELD MAPPED INTERACTIONS

In this section, we apply the mean-field mapping to the specific cases, and we present the results for the pseudopotentials calculated from the different mean-field mappings. In addition to the short-range four-body interaction ($V_6 = 1, V_{l\neq 6} = 0$), we also consider the case of the longer-range four-body interaction ($V_8 = 1, V_{l\neq 8} = 0$). The latter is a hollow-core four-body interaction. Analogous hollow-core two- and three-body interactions have been found to be useful in studies of fractional quantum Hall states such as at $\nu = 4/11$ [23–25].

A. $\mathrm{MF}^2_{\langle c^\dagger c\rangle}$

As described before, $\mathrm{MF}_{\langle c^\dagger c\rangle}$ applied twice ($\mathrm{MF}^2_{\langle c^\dagger c\rangle}$) maps the four-body interaction to a two-body interaction. Two-body pseudopotentials are extracted using a direct diagonalization of a system of only two particles. Since Hilbert space for such systems is relatively small, it is possible to calculate coefficients for systems with large $2Q$. In Table I we present values of the pseudopotentials for the two largest systems that we have studied. Irrespective of system size, only the first three allowed two-body pseudopotentials are nonzero in the mean-field mapping of the $V_6 = 1, V_8 = 0$ interaction, and only the first four allowed two-body pseudopotentials are found to be nonzero in the mean-field mapping of the $V_6 = 0, V_8 = 1$ interaction.

TABLE I. Two-body pseudopotentials obtained by mean-field mapping $\mathrm{MF}^2_{\langle c^\dagger c\rangle}$ of the four-body interactions. Data are presented for the two largest systems studied, and for the two types of interactions first a short-range repulsion where a single four-body pseudopotential V_6 is nonzero and second a longer-range interaction with only V_8 being nonzero.

	$V_6^{(4)}=1,\ V_8^{(4)}=0$		$V_6^{(4)}=0,\ V_8^{(4)}=1$	
$V_n^{(2)}\downarrow$	$2Q=60$	$2Q=62$	$2Q=60$	$2Q=62$
V_1	7.01312	7.02084	6.14707	6.15839
V_3	3.46795	3.46901	2.68532	2.68741
V_5	1.26689	1.26630	1.04738	1.04576
V_7	0	0	1.81347	1.81125

The values of the two-body pseudopotentials for smaller systems are presented in Figs. 2 and 3. The data allow an extrapolation to the $2Q\to\infty$ limit using a simple function $V(2Q)=a+\frac{b}{c-2Q}$. Uncertainties of the coefficients a,b,c are very small; we present them in Table V in Appendix A. For the short-range four-body repulsion, the two-body pseudopotentials in the $2Q\to\infty$ limit have values $V_1:V_3:V_5=7.249\,75:3.500\,13:1.249\,96\approx 5.8:2.8:1$. This mean-field two-body interaction is identical to the optimal two-body interaction for the $k=3$ Read-Rezayi state obtained in Ref. [10], wherein the authors had studied systems of sizes up to $N=21$ and $2Q=32$ and found the ratio to be $6:3:1$. At the same flux, the mean-field approximation gives pseudopotentials in the ratio $5.3:2.6:1$. When comparing the pseudopotentials, we note that the numerical search for the optimal interaction (Ref. [10]) was performed on a finite grid in the parameter space, which is expected to result in finite error bars on the optimal pseudopotentials. A similar mean-field approximation to the $V_8=1$ interaction gives the ratios $V_1:V_3:V_5:V_7\approx 6.5:2.75:1:1.75$ in the large-$2Q$ limit (Fig. 3).

Linearity of the mean-field mapping $\mathrm{MF}(H_1)+\mathrm{MF}(H_2)=\mathrm{MF}(H_1+H_2)$ implies that the mean-field pseudopotentials of a four-body interaction with $V_6=A, V_8=B$ can be obtained as the corresponding linear combination of the mean-field pseudopotentials of $V_6=1, V_8=0$ and $V_8=1, V_6=0$ given in the previous tables.

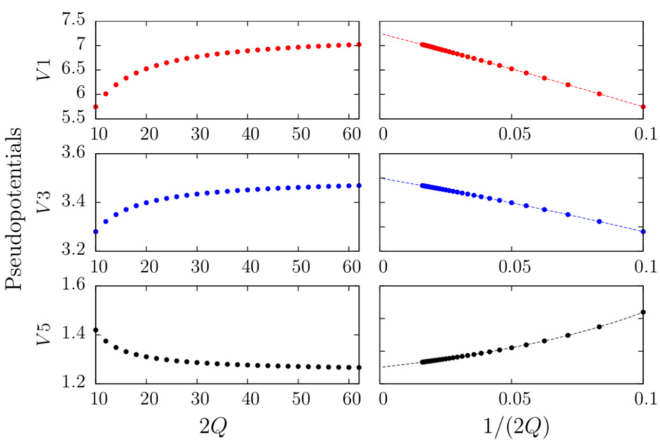

FIG. 2. Left: Two-body pseudopotentials obtained by the mapping $\mathrm{MF}^2_{\langle c^\dagger c\rangle}$ of four-body short-range repulsion with $V_6=1, V_8=0$. Right: Same information shown as a function $1/2Q$ to show convergence to the values in the $2Q\to\infty$ limit. The dotted lines indicate the fitting function $a+b/(c-2Q)$.

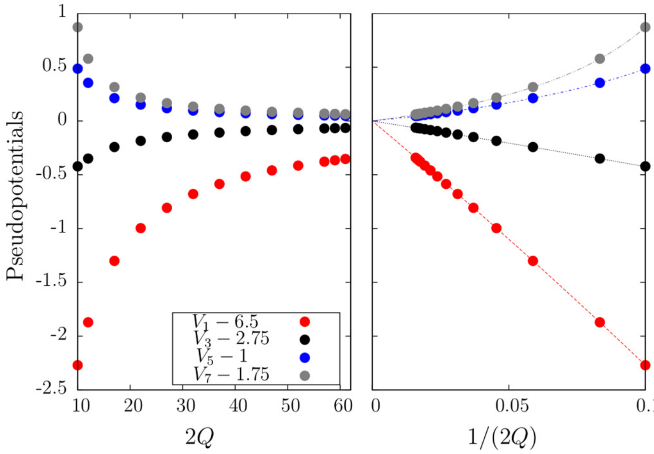

FIG. 3. Left: Two-body pseudopotentials obtained by the mapping $\mathrm{MF}^2_{\langle c^\dagger c\rangle}$ of longer-range four-body repulsion ($V_8=1, V_6=0$). Right: Same information plotted as a function of $1/2Q$. The vertical axis shows the deviation from the values in the $2Q\to\infty$ limit. The dotted lines indicate the fitting function $a+b/(c-2Q)$.

B. $\mathrm{MF}_{\langle c^\dagger c\rangle}$

When the mean-field mapping $\mathrm{MF}_{\langle c^\dagger c\rangle}$ is applied to a four-body interaction only once, we obtain a three-body interaction. The three-body pseudopotentials obtained by diagonalizing a system of three particles are presented in Table II.

The values of the three-body pseudopotentials in the large $2Q$ limit can also be inferred using a fitting function $a+\frac{b}{c-2Q}$ (Figs. 4 and 5). For the values of the coefficients a,b,c and their dispersion, see Table V.

C. $\mathrm{MF}_{\langle c^\dagger c^\dagger cc\rangle}$

As described in Sec. II, one can directly map a four-body interaction to an approximate two-body interaction by replacing the composite quartic operator $c_i^\dagger c_j^\dagger c_k c_l$ by the ground-state expectation values. Unlike the previous two cases, where the only information required to define the mapping came

TABLE II. Three-body pseudopotentials obtained by reduction of four-body interactions to three body using the mapping $\mathrm{MF}_{\langle c^\dagger c\rangle}$. Data are presented for the two largest systems and for the two types of interactions: short-range repulsion ($V_6=1, V_8=0$) and the longer-range repulsion ($V_6=0, V_8=1$).

	$V_6^{(4)}=1,\ V_8^{(4)}=0$		$V_6^{(4)}=0,\ V_8^{(4)}=1$	
$V_n^{(3)}\downarrow$	$2Q=47$	$2Q=52$	$2Q=47$	$2Q=52$
V_3	3.10234	3.10812	1.9913	2.0027
V_5	1.19654	1.19537	0.6742	0.6726
V_6	0.98568	0.98589	0.0783	0.0770
V_7	0	0	1.6232	1.6187
V_8	0	0	0.9067	0.9084

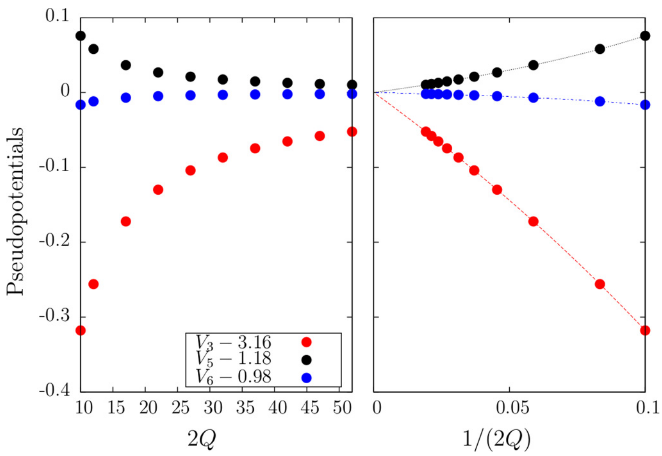

FIG. 4. Three-body pseudopotentials obtained by $\mathrm{MF}_{\langle c^\dagger c\rangle}$ mapping of the four-body short-range repulsion ($V_6 = 1, V_8 = 0$) to three-body interaction. The dotted lines indicate the fitting function $a + b/(c - 2Q)$.

from an assumption of homogeneity and rotational symmetry of the ground state (which implied $\langle c_i^\dagger c_k\rangle \propto \delta_{ik}$), a definition of the $\mathrm{MF}_{\langle c^\dagger c^\dagger cc\rangle}$ mapping requires more specific knowledge of the many-body state in which $\langle c_i^\dagger c_j^\dagger c_k c_l\rangle$ is calculated. This prevents us from implementing and exploring this mean-field calculation for systems larger than $2Q = 27$.

We estimated $\langle c_i^\dagger c_j^\dagger c_k c_l\rangle$ in the incompressible ground state ($k = 3$ Read-Rezayi state) of the short-range four-body interaction ($V_6 = 1, V_8 = 0$) of $N = 15$ and 18 particles at flux $2Q = 22$ and 27, respectively. From the two-body interaction obtained from this approximation, the pseudopotential can again be estimated from the energies of two particles. Table III contains pseudopotentials of reduced interaction for the largest systems that we studied. The pseudopotentials at $2Q = 27$ occur in the ratio $V_1 : V_3 : V_5 = 5.8 : 3.0 : 1$ matching closely with the results of Ref. [10].

The longer-range four-body interaction ($V_8 = 1, V_6 = 0$) does not produce an incompressible state at $2Q = \frac{5}{3}N - 3$ at every N. In the absence of a gapped ground state, it is not clear that such a mean-field approximation will work. Nevertheless, a mean-field approximation can still be constructed for the $V_8 = 1$ interaction using the correlations calculated from its ground state in the $L = 0$ sector. Table IV presents two-body pseudopotentials of mean-field reduction of this interaction.

Note that since the correlations used to reduce the interactions in the two cases (Tables III and IV) are not the same, the linearity property (which can be applied in the previous two cases $\mathrm{MF}_{\langle c^\dagger c\rangle}$ and $\mathrm{MF}^2_{\langle c^\dagger c\rangle}$) does not apply here.

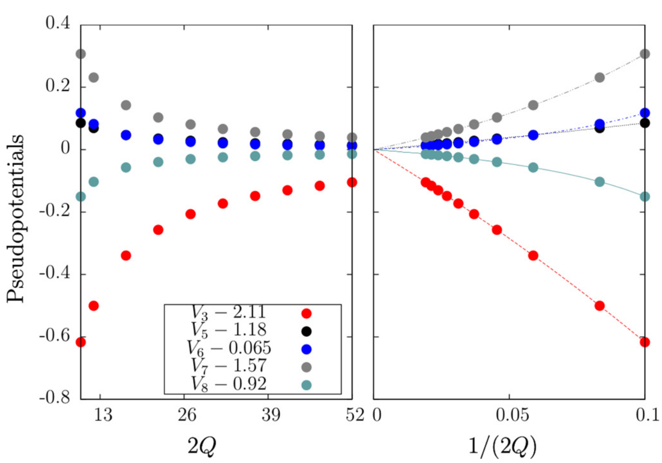

FIG. 5. Three-body pseudopotentials obtained by $\mathrm{MF}_{\langle c^\dagger c\rangle}$—reduction of longer-range four-body interaction ($V_8 = 1, V_6 = 0$) to three-body interaction. The dotted lines indicate the fitting function $a + b/(c - 2Q)$.

TABLE III. Two-body pseudopotentials from the mapping $\mathrm{MF}_{\langle c^\dagger c^\dagger cc\rangle}$ of the short-range four-body pseudopotential ($V_6 = 1, V_8 = 0$). The correlations $\langle c^\dagger c^\dagger cc\rangle$ are taken from the Read-Rezayi state.

Four-body $V_6 = 1$; Correlation from the ground state of $V_6 = 1$		
$V_n^{(2)}$	$2Q = 22$	$2Q = 27$
V_1	2.48198	2.45339
V_3	1.25684	1.22514
V_5	0.43826	0.41369

D. Comparison of pseudopotentials

Figure 6 shows a comparison of the two-body pseudopotentials obtained from the mean-field approximation of short-range four-body repulsion, Coulomb repulsion, and the two-body interaction obtained from the mean-field approximation of short-range three-body repulsion (described in Ref. [12]) all normalized such that $V_1^{(2)} = 1$.

IV. MEAN-FIELD APPROXIMATION AND PARTICLE-HOLE ANTISYMMETRIZATION AND SYMMETRIZATION

In this section, we will explore the connection between the mean-field approximation and the particle-hole symmetrization and antisymmetrization of multibody interactions. It was found in Ref. [12] that the spectrum of the mean-field approximation of the short-range three-body interaction matches exactly with the spectra of the interaction obtained by particle-hole symmetrizing the short-range three-body interaction [20], suggesting that the two methods result in the same interaction. We will show here that the mean-field approximation to a *general* three-body Hamiltonian is identical (up to additive constant chemical potential terms and overall scaling

TABLE IV. Two-body pseudopotentials from the mapping $\mathrm{MF}_{\langle c^\dagger c^\dagger cc\rangle}$ of the longer-range four-body interaction ($V_8 = 1, V_6 = 0$). The correlations $\langle c^\dagger c^\dagger cc\rangle$ are taken from the lowest energy $L = 0$ state of the same four-body interaction.

Four-body $V_8 = 1$; Correlation from ground state of $V_8 = 1$		
$V_n^{(2)}$	$2Q = 22$	$2Q = 27$
V_1	2.00931	1.91807
V_3	1.12664	1.22146
V_5	0.41725	0.38017
V_7	0.6585	0.56026

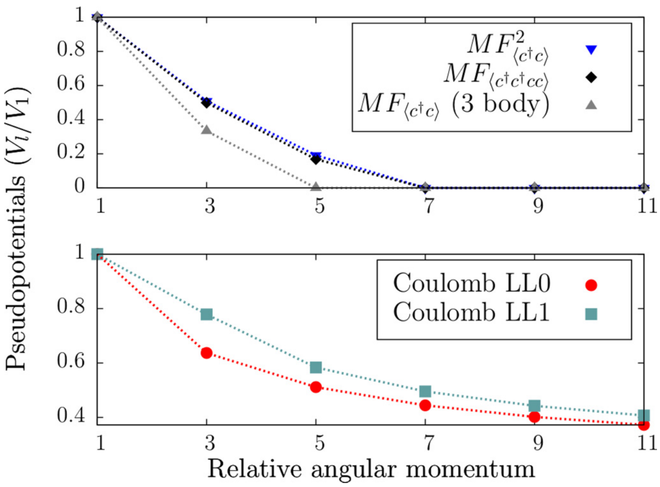

FIG. 6. Comparison of two-body pseudopotentials for the following: Coulomb interaction, $\mathrm{MF}^2_{\langle c^\dagger c\rangle}$ and $\mathrm{MF}_{\langle c^\dagger c^\dagger cc\rangle}$, obtained from four-body short-range repulsion ($2Q = 27$), the thermodynamic limit of pseudopotentials obtained from the mean-field approximation $\mathrm{MF}_{\langle c^\dagger c\rangle}$ of three-body short-range repulsion.

factors) to the particle-hole symmetrization of the same. As the algebra involved is the same as in Wick's theorem, we can immediately generalize the ideas to the case of four-body interactions.

A. Symmetrization of the three-body interaction and its mean-field approximation

A general three-body interaction can be written as

$$\mathcal{H}^{(3)} = \frac{1}{3!3!}\sum_{\mathbf{p},\mathbf{q}} V_{\mathbf{p};\mathbf{q}} c^\dagger_{p_3} c^\dagger_{p_2} c^\dagger_{p_1} c_{q_1} c_{q_2} c_{q_3}, \tag{7}$$

where $\mathbf{p} \equiv (p_1, p_2, p_3)$, $\mathbf{q} \equiv (q_1, q_2, q_3)$, and the sum is over $-Q \leqslant p_i, q_i \leqslant Q$ without any constraints on the ordering inside $\mathbf{p}$,

$$V_{\mathbf{p},\mathbf{q}} = \langle p_1 p_2 p_3 | \mathcal{H}^{(3)} | q_1 q_2 q_3 \rangle.$$

The particle-hole conjugation of the interaction is given by

$$\overline{\mathcal{H}}^{(3)} = \frac{1}{3!3!}\sum_{\mathbf{p},\mathbf{q}} V_{\mathbf{p},\mathbf{q}} c_{p_3} c_{p_2} c_{p_1} c^\dagger_{q_1} c^\dagger_{q_2} c^\dagger_{q_3}. \tag{8}$$

Shifting the creation operator to the right using the commutation relations reveals a relation between $\mathcal{H}^{(3)}$ and its particle-hole conjugate,

$$\overline{\mathcal{H}}^{(3)} = \frac{1}{3!3!}\sum_{\mathbf{p},\mathbf{q}} V_{\mathbf{p},\mathbf{q}}\big[C^{(0)}_{\mathbf{p},\mathbf{q}} + C^{(2)}_{\mathbf{p},\mathbf{q}} + C^{(4)}_{\mathbf{p},\mathbf{q}}\big] - \mathcal{H}^{(3)}. \tag{9}$$

Here

$$C^{(0)}_{\mathbf{p},\mathbf{q}} = \frac{1}{3!}\sum_{Q,P\in S_3} (-1)^{PQ} \delta_{Q(q_1)P(p_1)} \delta_{Q(q_2)P(p_2)} \delta_{Q(q_3)P(p_3)},$$

$$C^{(2)}_{\mathbf{p},\mathbf{q}} = -\frac{1}{2!}\sum_{Q,P\in S_3} (-1)^{PQ} \delta_{Q(q_1)P(p_1)} \delta_{Q(q_2)P(p_2)} c^\dagger_{Q(q_3)} c_{P(p_3)},$$

$$C^{(4)}_{\mathbf{p},\mathbf{q}} = \frac{1}{2!2!}\sum_{P,Q\in S_3} (-1)^{PQ} \delta_{Q(q_1)P(p_1)} c^\dagger_{Q(q_3)} c^\dagger_{Q(q_2)} c_{P(p_2)} c_{P(p_3)},$$

where S_3 is the permutation group over three objects. These are precisely the terms that arise when Wick's theorem is used to relate the particle-hole conjugate interaction [Eq. (8)] to the normal ordered form, with the contraction being equivalent to setting $\langle c^\dagger_i c_j\rangle$ to be δ_{ij}.

The first term $C^{(0)}$ gives a constant contribution to the right-hand side of Eq. (9). The second term arising from $C^{(2)}$ is nonzero only when a pair of entries in $\mathbf{p}$ match with a pair in $\mathbf{q}$. Considering that $V_{\mathbf{p},\mathbf{q}}$ is nonzero only when $\sum p_i$ match with $\sum q_i$, we find that the $C^{(2)}$ is proportional to

$$\sum_{p=-Q}^{Q} A_{pp} c^\dagger_p c_p,$$

where

$$A_{pp} = \sum_{p_1,q_1,p_2,q_2=-Q}^{Q} V_{(p_1,p_2,p);(q_1,q_2,p)} \delta_{p_1 q_1} \delta_{p_2 q_2}.$$

It can be seen that A_{pp} is independent of p. Rotational symmetry of the interaction implies that the elements $V_{\mathbf{p},\mathbf{q}}$ are linear combinations of projectors onto fixed angular momentum subspaces, i.e.,

$$V_{\mathbf{p},\mathbf{q}} = \sum_L a_L \big[P^{(3)}_L\big]_{\mathbf{p},\mathbf{q}}.$$

Therefore, $V_{\mathbf{p},\mathbf{q}}$ is a rotationally invariant tensor, i.e., invariant under the rotation R [written in the $(2Q+1)$-dimensional representation],

$$V_{\mathbf{p},\mathbf{q}} = \sum_{\dot{\mathbf{p}},\dot{\mathbf{q}}} R_{p_1\dot{p}_1} R_{p_2\dot{p}_2} R_{p_3\dot{p}_3} V_{\dot{\mathbf{p}},\dot{\mathbf{q}}} \bar{R}_{\dot{q}_1 q_1} \bar{R}_{\dot{q}_2 q_2} \bar{R}_{\dot{q}_3 q_3}. \tag{10}$$

Similarly $\delta_{p_1 q_1}\delta_{p_2 q_2}$ is a rotationally invariant tensor. So $A_{p_3 q_3}$ obtained by contracting the four indices p_1, p_2, q_1, q_2 of the two tensors is also symmetric, i.e., $A_{pq} = \sum_{\dot{p}\dot{q}} R_{p\dot{p}} \bar{R}_{\dot{q}q} A_{\dot{p}\dot{q}}$. Since there exists some rotation R, which takes an angular momentum p to another angular momentum p', we have that $A_{pp} = A_{p'p'}$. This implies that the term $C^{(2)}$ is simply a uniform chemical potential shift.

Finally, the term $C^{(4)}$ can be shown to be proportional to the mean-field approximation of the three-body interaction. Therefore, we have that up to multiplicative and additive constants,

$$\mathcal{H}^{(3)} + \overline{\mathcal{H}}^{(3)} \propto \mathrm{MF}_{\langle c^\dagger c\rangle}(\mathcal{H}^{(3)}). \tag{11}$$

Therefore, the spectra of the particle-hole symmetrization of the three-body Hamiltonian [18,20], and the mean-field approximation of the same three-body Hamiltonian, are identical [12].

B. Particle-hole antisymmetrization of the four-body interaction

In this section, we ask if the relation shown in the previous section in the context of three-body interactions generalizes to the context of the four-body interaction, and we show that such a simple exact relation does not exist between the particle-hole symmetrization and the mean-field Hamiltonians. Consider the expansion of the particle-hole conjugate of the four-body interaction in terms of a sequence of normal ordered operators,

$$\overline{\mathcal{H}}^{(4)} = \mathcal{H}^{(4)} - \mathcal{H}^{(4\to 3)} + \mathcal{H}^{(4\to 2)} - \mathcal{H}^{(4\to 1)} + \mathcal{H}^{(4\to 0)}, \quad (12)$$

where the first term is the four-body interaction and the terms $\mathcal{H}^{(4\to n)}$ for $n = 3, 2, 1, 0$ are obtained under a sequence of applications of $\mathrm{MF}_{\langle c^\dagger c\rangle}$. Equivalently, these are the terms obtained after one, two, three, and four contractions $\langle c^\dagger c\rangle \propto \delta_{ij}$. $\mathcal{H}^{(4\to 0)}$ is a constant shift. As discussed in the previous section, rotational invariance implies that the $\mathcal{H}^{(4\to 1)}$ is also a constant chemical potential shift.

The above expression tells us that unlike the case of the three-body interaction, it is the particle-hole–antisymmetrization of the four-body interaction that contains fewer-body interactions. Up to constant shifts, we get the following results:

$$\mathcal{H}^{(4)} - \overline{\mathcal{H}}^{(4)} = \mathcal{H}^{(4\to 3)} - \mathcal{H}^{(4\to 2)},$$

$$\mathcal{H}^{(4)} + \overline{\mathcal{H}}^{(4)} = 2\mathcal{H}^{(4)} - \mathcal{H}^{(4\to 3)} + \mathcal{H}^{(4\to 2)},$$

where $\mathcal{H}^{(4\to 3)} \propto \mathrm{MF}_{\langle c^\dagger c\rangle}(\mathcal{H}^{(4)})$ and $\mathcal{H}^{(4\to 2)} \propto \mathrm{MF}^2_{\langle c^\dagger c\rangle}(\mathcal{H}^{(4)})$. Neither particle-hole symmetrization nor antisymmetrization produces a simple interaction that can be expanded in terms of positive pseudopotentials. In general, for (even) odd n, the particle-hole (anti)symmetrization of the n-body interaction produces an interaction that can be interpreted as a sum of $n-1$ and fewer-body interactions, albeit with some negative pseudopotentials.

V. NUMERICAL TESTS OF THE MEAN-FIELD APPROXIMATIONS

In this section, we present the results of numerical tests of the three mean-field approximations described in Sec. II. In particular, we focus on the states at filling fraction $\nu = 3/5$ at which the short-range four-body interaction with $V_6 = 1$ (other pseudopotentials are zero) produces an incompressible ground state. We compare the spectra of the approximations with that of the original four-body interaction. Since the mean-field approximations produce the same two-body interaction obtained in Ref. [10], the numerical tests given below extend the numerical tests presented there.

A. Spectrum on the sphere

An incompressible ground state representing a filling fraction of 3/5 is produced by the short-range four-body interaction in a system of N (a multiple of 3) electrons on a sphere pierced by $2Q = 5N/3 - 3$ radial flux quanta. This incompressible state corresponds to the $k = 3$ Read-Rezayi state [7]. Figure 7 (top left) shows the spectrum of such a system of $N = 18$ particles. This incompressible state for a system of N particles can be written as [7]

$$\mathcal{A}\left[\Psi_{\frac{1}{3}}(\mathbf{z})\Psi_{\frac{1}{3}}(\mathbf{w})\Psi_{\frac{1}{3}}(\mathbf{r})\prod_{i,j=1}^{N/3}(z_i - r_j) \times \prod_{i,j=1}^{N/3}(z_i - w_j)\prod_{i,j=1}^{N/3}(r_i - w_j)\right], \quad (13)$$

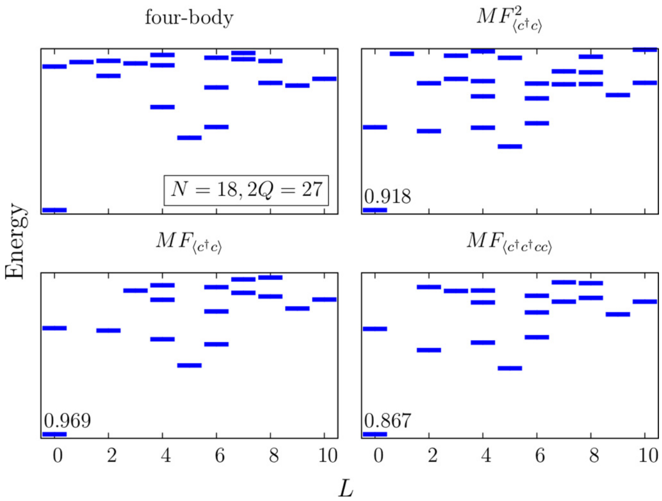

FIG. 7. Spectra of a four-body short-range repulsion Hamiltonian and its mean-field approximations at a filling factor $\nu = 3/5$, $N = 18$, $2Q = 27$. The numbers next to the ground states show the overlap of the corresponding state with the ground state of the four-body interaction.

where $\mathbf{z}, \mathbf{w}, \mathbf{r}$ are partitions into three equal parts of the N coordinates. The function $\Psi_{\frac{1}{3}}$ is the Laughlin state at filling fraction 1/3. The symbol $\mathcal{A}$ indicates antisymmetrization over N coordinates and ensures that the function represents a wave function of N indistinguishable particles. The function is expressed in the language of disk geometry, but it can be straightforwardly mapped to the spherical geometry using a stereographic projection.

Just above the gapped ground state is a neutral mode whose wave function corresponds to the one in which one of the partitions $\Psi_{\frac{1}{3}}$ has a neutral excitation [9,26,27]. Using this construction, the allowed quantum numbers of the neutral mode can be inferred to be $0 < L \leqslant N/3$. In Fig. 7 (top left), the neutral mode can be seen to extend up to an angular momentum $L = 6$ as expected, however the mode merges into the bulk spectrum at low angular momenta.

The spectrum of the two-body interaction obtained using the mean-field approximation $\mathrm{MF}^2_{\langle c^\dagger c\rangle}$ is shown in Fig. 7 (top right). The spectrum contains a unique $L = 0$ ground state with a high overlap with the Read-Rezayi state. A mode of excitations can be seen above this whose counting matches at larger angular momentum but appears to differ at lower angular momenta. Note that for neutral excitations (which are made of a quasiparticle-quasihole pair), lower angular momenta correspond to states in which the quasiparticle and quasihole are close to each other. Relative agreement in the spectra as angular momentum increases suggests that this mean-field approximation reproduces the right long-distance physics. The three-body interaction obtained from $\mathrm{MF}_{\langle c^\dagger c\rangle}$ also produces an incompressible state [Fig. 7 (bottom left)] with a high overlap with the Read-Rezayi state as well as a neutral mode with the right quantum numbers. The spectra of the two-body interaction obtained using $\mathrm{MF}_{\langle c^\dagger c^\dagger cc\rangle}$, shown in Fig. 7 (bottom right), are qualitatively similar to the spectra of the two-body interaction obtained from $\mathrm{MF}^2_{\langle c^\dagger c\rangle}$. Note that the pseudopotentials depend on $2Q$. The calculations presented

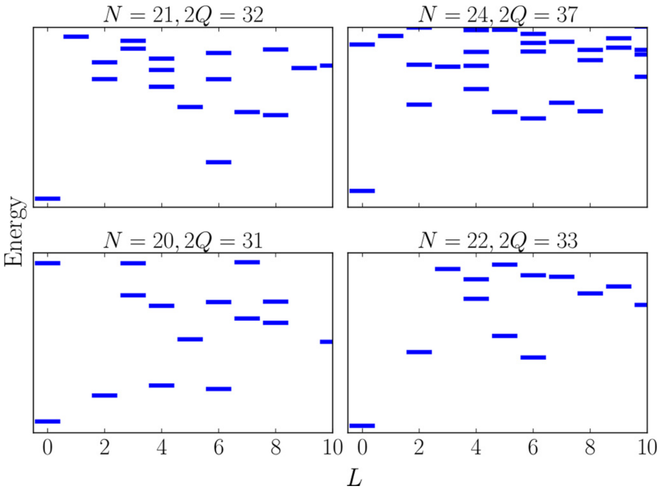

FIG. 8. The spectra of the mean-field approximation $\mathrm{MF}^2_{\langle c^\dagger c\rangle}$ to the short-range four-body Hamiltonian with nonzero pseudopotential for V_6 at $N = 21, 2Q = 32$; $N = 24, 2Q = 37$; $N = 20, 2Q = 31$; and $N = 22, 2Q = 33$.

here use mean-field pseudopotentials at the respective fluxes and not the ones inferred for the thermodynamic limit.

The results presented in Fig. 7 are for the largest system in which all interactions were studied. Though diagonalization of four-body interaction in larger systems is not easy, the quantum numbers of the low-energy states can be inferred from the trial wave-function approach discussed above. The mean-field two-body interaction can be diagonalized in bigger systems, and the low-energy quantum numbers can be compared with those from the trial wave functions. Figure 8 (top left) shows the spectrum of the two-body interaction obtained using $\mathrm{MF}^2_{\langle c^\dagger c\rangle}$ in a system $N = 21, 2Q = 32$. The interaction again produces a homogeneous incompressible ground state and a neutral mode. However, we find that the neutral mode ends at angular momentum $L = 8$ instead of $L = 7$. The top-right panel shows the spectrum of the mean-field two-body interaction in a larger system $N = 24, 2Q = 37$. Here we find that the neutral mode counting matches with the predicted angular momentum of $L = 8$.

The structure of the neutral mode is indirectly encoded in the ground-state pair correlation functions [28], and therefore we expect that these functions should also be reproduced by the approximate Hamiltonians. Figure 9 shows the pair correlation functions in the incompressible ground state of the four-body interaction as well as in the ground states of the different approximate Hamiltonians shown in Fig. 7.

Figure 10 shows the spectrum of a system $N = 17, 2Q = 26$, which has one electron and a flux less than that in the incompressible state. The low-energy spectrum arises from two quasiholes of the Read-Rezayi state. The wave functions at low energies can be understood in terms of a three-partition structure similar to that in Eq. (13), wherein the three partitions now contain six, six, and five electrons in the states $\Psi_{\frac{1}{3}}$, $\Psi_{\frac{1}{3}}$, and $\Psi^{2\mathrm{qh}}_{\frac{1}{3}}$; the two quasiholes correspond to two quasiholes in the Laughlin state in one of the three partitions [26]. Using this structure of the wave function, the angular momenta of the low-energy states can be calculated to be

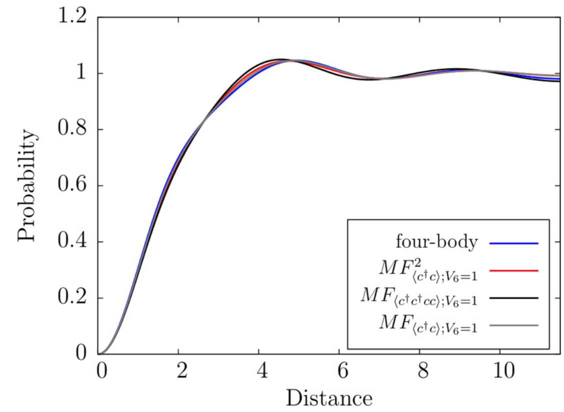

FIG. 9. Pair correlation functions calculated from the incompressible ground state at flux $2Q = 5N/3 - 3$ of the short-range four-body interaction as well as its mean-field approximations.

$L = 1, 3, 5$ and can be verified in the exact spectrum. Here we find that the spectra of all mean-field approximations closely resemble the spectrum of the four-body interaction.

Figure 8 (bottom left) shows the spectrum of the system $N = 20, 2Q = 31$, which is again obtained by removing an electron and a flux from the incompressible system at $N = 21, 2Q = 32$. This can again be understood as a similar two-quasihole state, and the angular momenta of the low-energy states can be calculated to be $L = 0, 2, 4, 6$. This exactly matches what is seen in the spectra of the mean-field two-body Hamiltonian.

Figure 11 shows the spectra for a system at $N = 19, 2Q = 28$, with one electron and one flux more than in the incompressible state. The low-energy spectrum is expected to correspond to a system containing a pair of quasiparticles. The wave function can be understood as containing three partitions [similar to Eq. (13)] of six, six, and seven particles in the states $\Psi_{\frac{1}{3}}$, $\Psi_{\frac{1}{3}}$, and $\Psi^{2\mathrm{qp}}_{\frac{1}{3}}$, the two quasiparticles being in the last partition. The quantum numbers of the two-quasiparticle

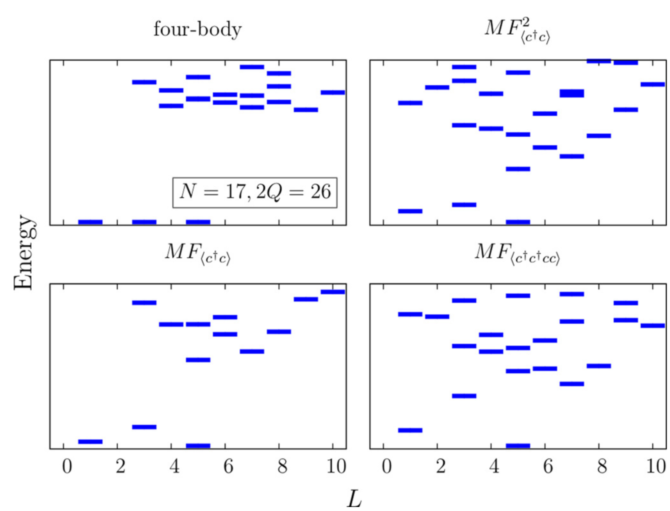

FIG. 10. The spectra of short-range four-body repulsion ($V_6 = 1, V_8 = 0$) and its mean-field approximation at $N = 17, 2Q = 26$ (corresponding to two quasihole states).

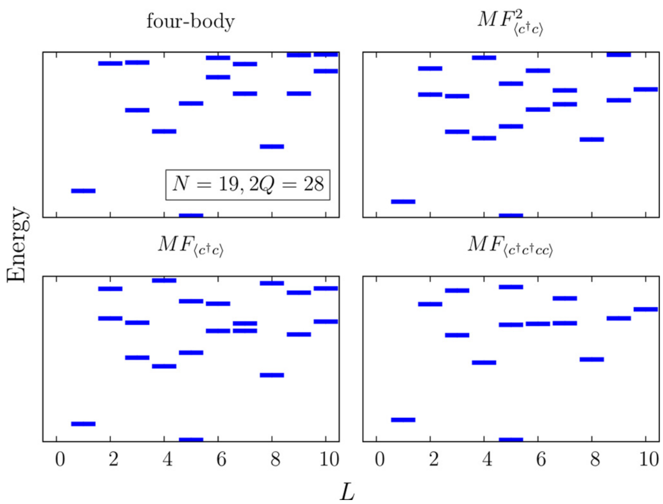

FIG. 11. The spectra of short-range four-body repulsion ($V_6 = 1, V_8 = 0$) for a system of $N = 19, 2Q = 28$ (corresponding to two quasiparticles).

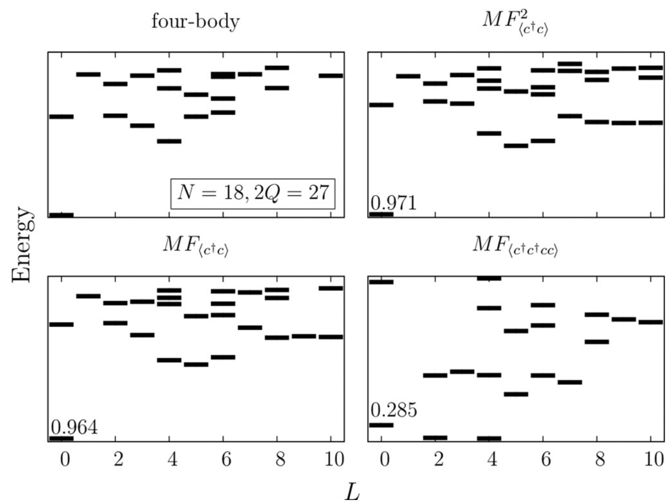

FIG. 12. The spectra of the longer-range four-body repulsion Hamiltonian with nonzero pseudopotential for $V_8 = 1$ alone and its mean-field spectra in a system of size $N = 18, 2Q = 27$.

state within this picture is then the same as the quantum numbers $L = 1, 3, 5$ of the two quasiparticles of the Laughlin state in the last partition. A clearly separated quasiparticle branch with this counting cannot be seen even in the original four-body interaction. Since the two quasiparticles are closer to each other with higher probability in the higher angular momentum states, it is expected that such a counting based on trial wave functions should work only in the low angular momentum limit. The spectra (Fig. 11) of all three mean-field approximations match with what is seen in the actual spectrum of the four-body interaction. Figure 8 (bottom right) shows the spectra of the mean-field two-body interaction in the next bigger system, where we expect a two-quasiparticle state. Based on the wave functions described above, the quantum numbers of the low-energy states are expected to be $L = 0, 2, 4, 6$. The quantum numbers in the spectra match with these numbers in the low angular momentum limit.

For completeness, we also explore the spectra of the mean-field approximation to the longer-range $V_8 = 1, V_6 = 0$ interaction. In general, the interaction does not produce an incompressible ground state in the same flux sector $2Q = \frac{5N}{3} - 3$ that we have studied. A gapped homogeneous state is produced in the specific case of $N = 18$ but not in the $N = 15$ case. Results for the spectrum of this interaction at $N = 18$ are shown in Fig. 12. In this case, the spectrum of the four-body interaction is closely reproduced by the $\mathrm{MF}_{\langle c^\dagger c\rangle}$ and $\mathrm{MF}^2_{\langle c^\dagger c\rangle}$ approximations but not by $\mathrm{MF}_{\langle c^\dagger c^\dagger cc\rangle}$. The difference in the spectrum is not surprising given that the two-body interactions obtained by the two methods (Tables IV and I) appear qualitatively different. For the case of $N = 15$, where there is no clear gap in the spectrum, we can still construct the approximate Hamiltonian using the correlation function in the ground state of the $L = 0$ sector. In this case, we find that $\mathrm{MF}_{\langle c^\dagger c^\dagger cc\rangle}$ produces a spectrum closer to the four-body interaction.

B. Spectrum on the torus

A characteristic signature of non-Abelian clustered states is the degeneracy of the ground state on manifolds of nonzero genus, the simplest being the torus. The Read-Rezayi state has a degeneracy on the torus of ten, five arising from center-of-mass translations [29] and the remaining two from the non-Abelian nature of the state [30]. Figure 13 shows the spectrum of the two-body interaction mapped onto rectangular torus of aspect ratios $R = 1.0$ and 0.9. The x axes show the relative

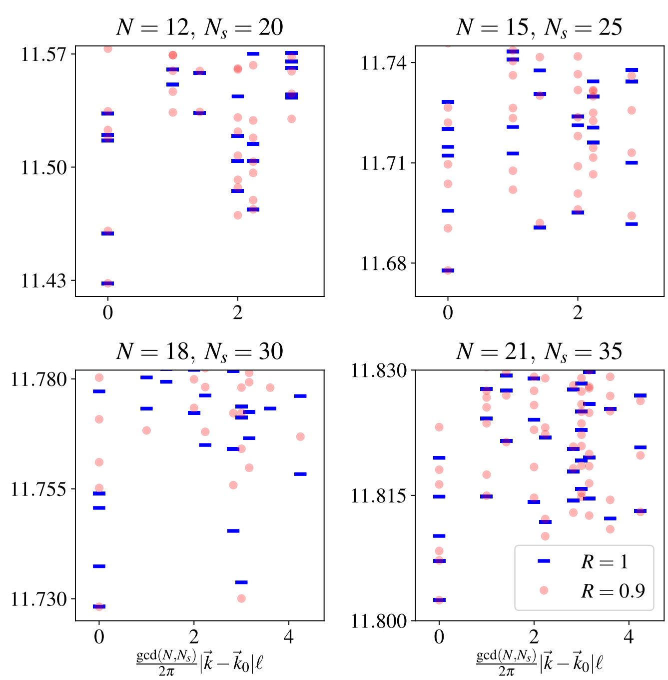

FIG. 13. Spectrum on the torus of the mean-field Hamiltonian (corresponding to the $\mathrm{MF}^2_{\langle c^\dagger c\rangle}$ pseudopotential on the sphere at the same flux $N_s \equiv 2Q$) plotted as a function of relative momenta $\boldsymbol{k}$ on the torus, shifted by the momentum of the ground state $\boldsymbol{k}_0 = (0, 0)/(\pi, \pi)$ for N odd/even. The blue dashes and red dots indicate the spectra for aspect ratios $R = 1, R = 0.9$ on the torus. The energy of the $R = 0.9$ has been shifted (not rescaled) such that their ground states match.

momenta $\boldsymbol{k}$ on the torus. Twofold-degenerate ground states are expected to occur for the four-body interaction at momenta $\boldsymbol{k}_0 = (0,0)/(\pi,\pi)$ for an even/odd number of particles. Note that since the interaction has been approximated, we do not expect exact double degeneracy in finite systems. While two low-energy states can be seen in the $\boldsymbol{k}-\boldsymbol{k}_0 = \boldsymbol{0}$ sectors in $N = 12, 15, 18$, this is not very clear in all system sizes. Moreover, for an even number of particles ($N = 12$ and 18), an additional low-energy state emerges at $\boldsymbol{k} = (0,\pi)$, possibly suggesting a competing anisotropic phase stabilized by the reduced rotational symmetry on the torus. The question of which phase is stable in the thermodynamic limit unfortunately cannot be addressed within the system sizes that we have considered.

VI. CONCLUSIONS

We have presented three approaches for approximating the four-body interaction to obtain fewer-body Hamiltonians, and we tested these approaches on systems around the flux value $2Q = \frac{5}{3}N - 3$, where the short-ranged four-body interaction produces a gapped ground state. Evidence from numerical diagonalization of finite-size systems suggests that the approximation schemes produce a good effective model of the physics of four-body interaction at filling fraction $\nu = 3/5$. The two-body pseudopotentials for approximation can be estimated to be close to $V_1 : V_3 : V_5 = 6 : 3 : 1$. A comparison with previous studies in Ref. [10] suggests that the obtained two-body approximations are indeed the optimal two-body interactions that produce the Read-Rezayi state. A similar approximation to the short-range three-body interaction produces the corresponding optimal interaction that approximates the low-energy physics around the Pfaffian state. The mean-field approximation of the three-body interaction is exactly the same as its particle-hole symmetrization up to constants, but such a relation is not true for the $n > 3$-body interactions. The two-body mean-field approximations, which seem to accurately reproduce the spectra, form only a part of the particle-hole symmetrization of the four-body interaction. It will be interesting to explore the importance of the particle-hole symmetry breaking and the symmetry-preserving corrections to the mean-field approximations. Study of the spectra in the torus geometry suggests that the mean-field interaction could be in close proximity to an anisotropic phase, making analysis of the spectra difficult within the accessible system sizes.

ACKNOWLEDGMENTS

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APPENDIX A: FITTING FUNCTIONS OF APPROXIMATE PSEUDOPOTENTIALS

The mean field pseudopotential can be calculated for relatively large systems in the case of $\mathrm{MF}^2_{\langle c^\dagger c\rangle}$ and $\mathrm{MF}_{\langle c^\dagger c\rangle}$.

TABLE V. The coefficients of a function $V_l^{(n)}(2Q) = a + \frac{b}{c-2Q}$, which is a best fit of $n = 2$- and 3-body pseudopotentials obtained from the mean-field approximation of a four-body interaction with a single nonzero four-body pseudopotential (V_6 or V_8).

	$\mathrm{MF}^2_{\langle c^\dagger c\rangle}(V_6^{(4)}=1, V_8^{(4)}=0)$		
$V_l^{(2)}$	a	b	c
$l=1$	7.24975	−14.0437	0.659577
$l=3$	3.50013	−1.88354	1.42402
$l=5$	1.24996	0.939755	4.45989
	$\mathrm{MF}^2_{\langle c^\dagger c\rangle}(V_6^{(4)}=0, V_8^{(4)}=1)$		
$V_l^{(2)}$	a	b	c
$l=1$	6.50009	−21.3186	0.608319
$l=3$	2.75182	−4.0066	0.552965
$l=5$	1.00101	2.62568	4.58301
$l=7$	1.75123	3.43484	6.05555
	$\mathrm{MF}_{\langle c^\dagger c\rangle}(V_6^{(4)}=1, V_8^{(4)}=0)$		
$V_l^{(3)}$	a	b	c
$l=3$	3.16047	−2.63259	1.71567
$l=5$	1.18517	0.494979	3.46206
$l=6$	0.987714	−0.085466	4.79941
	$\mathrm{MF}_{\langle c^\dagger c\rangle}(V_6^{(4)}=0, V_8^{(4)}=1)$		
$V_l^{(3)}$	a	b	c
$l=3$	2.10724	−5.28236	1.44103
$l=5$	0.658232	0.7258	1.51154
$l=6$	0.0652811	0.538105	5.42847
$l=7$	1.57983	1.8651	3.92187
$l=8$	0.922529	−0.64921	5.68722

The two-body psuedopotentials obtained from $\mathrm{MF}^2_{\langle c^\dagger c\rangle}$ are shown in Figs. 2 and 3 for the cases of four-body interactions with $V_6 = 1$ and $V_8 = 1$ respectively. Similarly, the three-body psuedopotentials obtained from $\mathrm{MF}_{\langle c^\dagger c\rangle}$ are shown in Figs. 4 and 5 for the cases of four-body interactions with $V_6 = 1$ and $V_8 = 1$ respectively.

Using the large range of the available pseudopotential data, we can obtain an approximate fitting function of the form $a + b/(c - 2Q)$ for each mean field pseudopotential. This could be useful in numerical studies at general values of $2Q$.

The coefficients of the fitting function for different cases are shown in Table V.

APPENDIX B: EXPANSION OF $\langle c^\dagger_{p_1} c^\dagger_{p_2} c_{q_2} c_{q_1}\rangle$ IN ANGULAR MOMENTUM CHANNELS

The mean field approximations are constructed by replacing certain simple composite operators in the four-body interaction by their expectation values estimated in the incompressible ground state of the four body interaction. It is interesting to ask how well some of the correlation functions such as the density-density correlations are reporoduced in the mean-field ground state. Figures 14 and 15 shows this information for the four-fermion correlation $M_{\mathbf{pq}} = \langle c^\dagger_{p_1} c^\dagger_{p_2} c_{q_1} c_{q_2}\rangle$. Thanks to the rotational symmetry, all the information in these correlations can be encoded in the eigenvalues of the correlation matrix M. The eigenvectors of M are the two particle angular momentum eigenstates.

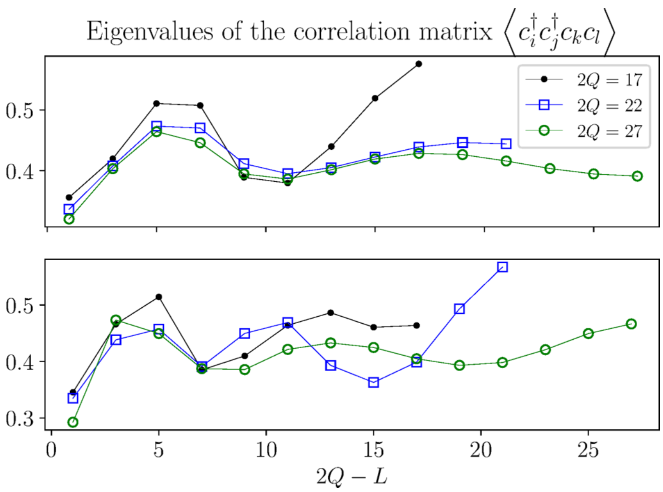

FIG. 14. Eigenvalues of the correlation matrix $M_{\mathbf{pq}} = \langle c^\dagger_{p_1} c^\dagger_{p_2} c_{q_2} c_{q_1} \rangle$ plotted as a function of the total angular momentum quantum number of the eigenvector. The top figure shows the correlation in the ground state of the short-range four-body interaction $V_8 = 0, V_6 = 1$ and the bottom figure shows the same for the ground state of the $L = 0$ sector ground state of the longer-range interaction $V_8 = 1, V_6 = 0$.

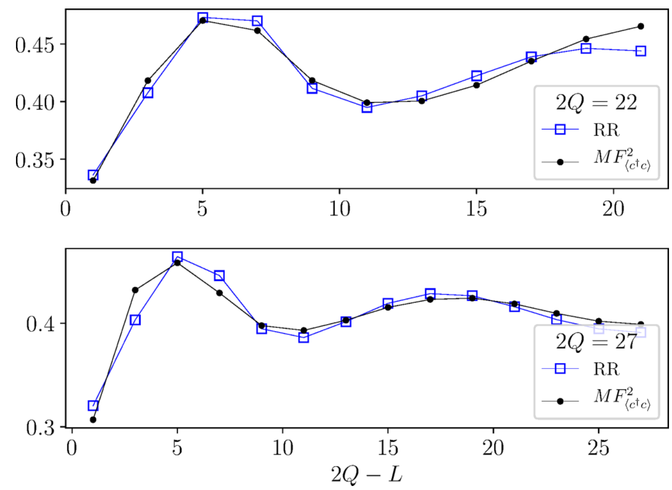

FIG. 15. Plot shows the eigenvalues of the correlation matrix as a function of the angular momentum similar to Fig. 14. The figure compares the correlations in the Read-Rezayi state with that in the ground state of the $\text{MF}^2_{\langle c^\dagger c\rangle}$ approximation of short-range interaction in two different system sizes $N = 15, 2Q = 22$ (top) and $N = 18, 2Q = 27$ (bottom).

Figure 14 shows these eigenvalues of M for the ground states of four-body interactions in different system sizes at $2Q = \frac{5N}{3} - 3$. Figure 15 shows that the eigenvalues of M in the ground state of the four-body interaction $V_6 = 1, V_{l\neq 6} = 0$ are closely reproduced in the ground state of its mean-field approximation.

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