Unpaired Composite Fermion, Topological Exciton, and Zero Mode

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The paired state of composite fermions is expected to support two kinds of excitations: vortices and unpaired composite fermions. We construct an explicit microscopic description of the unpaired composite fermions, which we demonstrate to be accurate for a 3-body model interaction and, possibly, adiabatically connected to the Coulomb solution. This understanding reveals that an unpaired composite fermion carries with it a charge-neutral "topological" exciton, which, in turn, helps provide microscopic insight into the origin of zero modes, fusion rules, and energetics.

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Should composite fermions (CFs) form a p-wave paired state, as has been proposed [1,2] for the mechanism of the fractional quantum Hall effect at $5/2$, analogy to superconductivity leads one to expect two kinds of excitations: vortices and unpaired CFs (UCFs) [3]. A number of their properties, predicted by the Bogoliubov–de Gennes treatment or conformal field theory [1,6,7], have been confirmed for excitations that are exact zero energy solutions of a model 3-body Hamiltonian H_3 (defined below), such as the quasihole vortices. The situation is less clear for other states, e.g., quasiparticle vortices and UCFs, as well as for the Coulomb interaction for which no accurate wave functions exist. Möller, Wójs, and Cooper [4] and Bonderson, Feiguin, and Nayak [5] have studied the UCF by exact diagonalization by considering states with an odd number (N) of composite fermions, which necessarily contain a composite fermion without a partner. This Letter presents a microscopic description of the UCF which reveals that the UCF carries with it what Hansson has termed a "topological" exciton [8,9]. That, in turn, yields a number of remarkable and physically transparent consequences for energetics, such as even-odd oscillations and zero modes [4,5,10], as well as fusion rules [1,6,7]. We present evidence that the microscopic description below is exceedingly accurate for H_3 and that its essential features carry over adiabatically to the Coulomb solutions.

We consider the following ansatz for the UCF state:

$$\Psi^{\mathrm{UCF}} = \mathcal{A}\,\phi^{\mathrm{CF}}(\{z_j\})\chi^{\mathrm{CF}}(\{w_k\})\prod_{j=1}^{M+1}\prod_{k=1}^{M}(z_j - w_k),$$

where the $N = 2M + 1$ composite fermions have been divided into two partitions, $\{z_1, z_2, \ldots, z_{M+1}\}$ and $\{w_1, w_2, \ldots, w_M\}$, occupying states $\phi^{\mathrm{CF}}(\{z_j\})$ and $\chi^{\mathrm{CF}}(\{w_k\})$, and the last term represents correlations between composite fermions in different partitions. The symbol $\mathcal{A}$ indicates antisymmetrization with respect to exchange of any two particles. Let us define the largest exponent of z_j in ϕ as $2Q_\phi$ and the largest exponent of w_k in χ as $2Q_\chi$, which are analogous to the flux (measured in units of the flux quantum $\phi_0 = hc/e$) in the spherical geometry for ϕ and χ. Because the net flux must be the same for all particles, we have $2Q = 2Q_\phi + M = 2Q_\chi + M + 1$ including the contribution from the cross factor. At $2Q = 2N - 3$, the flux value relevant for the $5/2$ state, ϕ is the $1/3$ state with a quasiparticle and χ is the $1/3$ state with a quasihole [Fig. 1(a)]. The unpaired CF state thus carries a charge-neutral exciton. This topological exciton is to be distinguished from the "ordinary" exciton [Fig. 1(b)] that contains a quasiparticle-quasihole pair within one partition; unlike in the ordinary exciton, the quasiparticle and the quasihole in the topological exciton cannot annihilate one another—they are part of the ground state continuum at odd N. Different placements of the quasiparticle and the quasihole [Fig. 1(a)] generate a basis for UCF states.

The form of Ψ^{UCF} is naturally motivated by the observation [2] that the Pfaffian wave function for even N can be written as the fully antisymmetrized "331" bilayer wave function of Halperin [11]. The conformal field theory construction of topological exciton by Hansson [8,9] is similar in spirit. These wave functions can be extended [12] to other states by starting with the more general bilayer wave functions of Scarola and Jain [13]. Wave functions of this form have also been motivated by Hermanns [14] and by Milovanović and Jolicœur [15] in a conformal field theory approach.

The ansatz Ψ^{UCF} signifies definite predictions, *with no free parameters*, for the quantum numbers of the low-energy states, their wave functions, and their energies, through which the theory opens itself to rigorous tests against exact results known for finite systems. The calculations below are performed in the spherical geometry in which the N electrons move on the surface of the sphere under the influence of a radial magnetic field. The total flux through this spherical surface is $2Qhc/e$. The wave

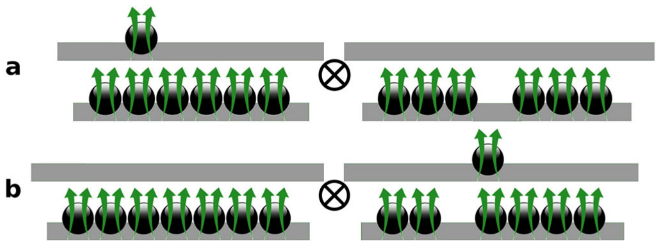

FIG. 1 (color online). Schematic depiction of (a) the unpaired composite fermion with a topological exciton and (b) an ordinary exciton. The composite fermions are shown as dots decorated with arrows, representing bound states of electrons and vortices. The left and right parts show the Λ level diagrams for composite fermions in the two partitions. The single CF in the otherwise empty Λ level is called a (CF) quasiparticle and the missing CF a (CF) quasihole, which, for the present case, have charge excess or deficiency of magnitude $e/4$ relative to the uniform ground state.

functions $\Psi^{\rm UCF}$ can be translated into the spherical geometry using standard methods. For the quasiparticle, Jain's CF wave function has been used [16,17]. The model 3-body interaction [2] is given by $H_3 = \sum_{i<j<k} P^{(3)}_{ijk}(3Q-3)$, where $P^{(3)}_{ijk}(L)$ projects the state of the three particles (i, j, and k) into the subspace of total orbital angular momentum L; the interaction penalizes the smallest approach of three particles. H_C denotes the Coulomb Hamiltonian in the second Landau level (LL). We find it convenient to express the UCF wave function in each angular momentum sector as a linear combination of exact eigenstates, $\psi^i_{\rm exact}$, of either H_3 or H_C in the same sector: $\Psi^{\rm UCF} = \sum_i c_i \psi^i_{\rm exact}$. By considering sufficiently many N-particle configurations, we obtain a system of linear equations that can be solved to obtain c_i. The energies and overlaps of $\Psi^{\rm UCF}$ are then evaluated straightforwardly [18].

The angular momentum of the CF quasiparticle in $\phi^{\rm CF}$ is $(N+1)/4$, and the angular momentum of the CF quasihole in $\chi^{\rm CF}$ is $(N-1)/4$, which gives the allowed angular momenta for their combination as $L = 1/2, \ldots, N/2$. It turns out, remarkably, that the state at $L = 1/2$ is exactly annihilated [19] upon antisymmetrization, thus leaving the UCF states at $L = 3/2, 5/2, \ldots, N/2$. The annihilation of the state at the smallest L is analogous to the annihilation of the $L = 1$ CF exciton of the fractional quantum Hall effect states at $\nu = n/(2n \pm 1)$, as noted by Dev and Jain [20].

The exact 3-body spectra (dots) are shown in the four upper panels (a)–(d) of Fig. 2 for several N. A low-energy branch of states (blue dots) is seen to be well separated from the continuum, and, with the exception of $L = 3/2$, the angular momenta of these states match nicely with the predicted values. The 3-body energies of $\Psi^{\rm UCF}$ and their overlaps with the corresponding exact eigenstates are also shown. The excellent agreement establishes the validity of

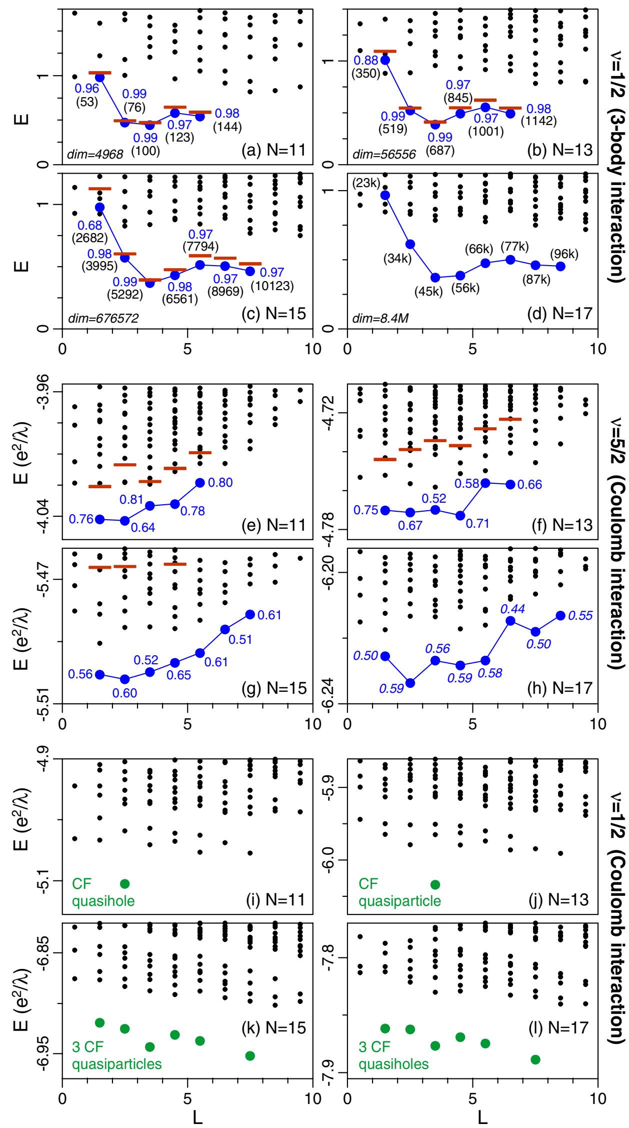

FIG. 2 (color online). Exact spectra (dots) for several particle numbers N at total flux $2Q = 2N - 3$ for the 3-body interaction (a)–(d), for the Coulomb interaction in the 2nd LL (e)–(h), and for Coulomb interaction in the lowest LL (i)–(l). L is the orbital angular momentum, and λ is the magnetic length. The fractional numbers near dots are overlaps with $\Psi^{\rm UCF}$ (which contains no adjustable parameters) in (a)–(c) and (e)–(g). The red dashes are the energies of $\Psi^{\rm UCF}$, not shown when they fall outside the frame [as in (g)]. In (h) we show overlaps with the 3-body eigenstates; their Coulomb energies all fall outside the frame. The total number of linearly independent states in each L sector is shown in brackets in (a)–(d), and "*dim*" indicates the dimension of the $L_z = 1/2$ basis used in exact diagonalization.

$\Psi^{\rm UCF}$ for H_3, with the exception of $L = 3/2$ where $\Psi^{\rm UCF}$ better describes an excited state; it should be noted that $L = 3/2$ represents the quasiparticle and quasihole at their shortest separation.

The exact spectra for H_C are shown in panels (e)–(h) of Fig. 2. These also contain a band of low-energy states

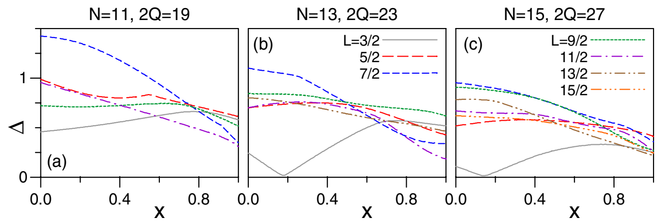

FIG. 3 (color online). Adiabatic evolution of the energy gap $\Delta(x)$ at each relevant value of L (defined as the separation between the two lowest eigenvalues at that L) as the interaction is varied from 3-body to 2nd LL Coulomb. The evolution Hamiltonian is $H(x) = (1 - x)H_3/\varepsilon_3 + xH_C/\varepsilon_C$ in which the energy scales ε_3 and ε_C are taken as the ordinary exciton energies at a large wave vector.

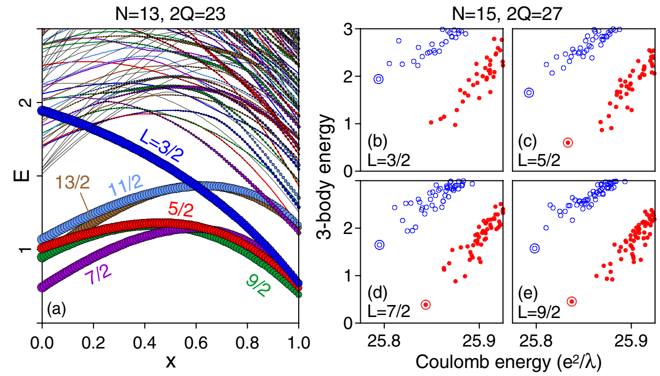

FIG. 4 (color online). (a) Evolution of the low-energy spectrum of $N = 13$ electrons with the Hamiltonian $H(x)$ of Fig. 3. Both constituents H_3 and H_C are measured from the uniform ground state energy (interpolated to an odd N). Colored lines with dots have been used for the angular momenta of the UCF, and the dot diameters give the overlaps with the corresponding 3-body eigenstate. Except for $L = 3/2$ (level crossing at $x \approx 0.2$), the UCF states at $x = 0$ appear adiabatically connected to the corresponding Coulomb states at $x = 1$. (b)–(e) Blue open circles show $\langle\Psi_C|H_3|\Psi_C\rangle$ for each 2nd LL Coulomb eigenstate Ψ_C as a function of its eigenenergy E_C, and the red dots show $\langle\Psi_3|H_C|\Psi_3\rangle$ for each 3-body eigenstate Ψ_3 as a function of its eigenenergy E_3. The encircled red and blue dots indicate that the lowest 3-body state has the lowest $\langle H_C\rangle$ and vice versa, supporting adiabatic continuity; the red dots at $L = 3/2$ provide an exception.

(blue dots) at $L = 3/2, 5/2, \ldots, N/2$, which is less well defined than the band for H_3, but the lowest state is well separated from the "continuum" at each L in this range. Unlike for H_3, the $L = 3/2$ state does not belong in the continuum. The overlaps of the Coulomb eigenstates with Ψ^{UCF} are moderate; this is to be expected because the overlap of the exact $5/2$ Coulomb ground state at even N with the Pfaffian wave function are also of similar level, and there is no reason why Ψ^{UCF} should do better than the Pfaffian wave function. We now ask if the Coulomb eigenstates are adiabatically connected to Ψ^{UCF}, as was argued to be the case by Storni, Morf, and Das Sarma [21] for the Pfaffian *ground* state for even N. We plot in Fig. 3 the evolution of the gap at each L as we vary the interaction from H_3 to H_C. The gap does not close for $L \geq 5/2$. There is an avoided level crossing at $L = 3/2$, but, interestingly, the Coulomb state is seen to be connected to the 3-body *excited* state that has largest overlap with the UCF wave function. These results suggest that the Coulomb eigenstates are adiabatically connected to Ψ^{UCF} for *all* relevant L. Further evidence for adiabatic continuity is presented in Fig. 4. We note that, as for the Pfaffian state, the agreement improves slightly (not shown) upon including finite thickness effects.

In contrast, the spectra for the *lowest* LL Coulomb state, shown in panels (i)–(l) of Fig. 2, are consistent with a system of weakly interacting composite fermions experiencing an effective flux $2|Q^*| = 2|Q - N + 1| = 1$. In the topmost partially filled Λ level shell, we have one CF hole with single particle angular momentum $l^* = 5/2$ for $N = 11$, one CF with $l^* = 7/2$ for $N = 13$, three CFs each with $l^* = 7/2$ for $N = 15$, and three CF holes each with $l^* = 7/2$ for $N = 17$. The allowed L values can be obtained from an elementary calculation and match precisely those seen in the exact spectra. The similarity of the lowest bands for 15 and 17 particles is striking; there is no symmetry in the electron problem which implies this result, but it is explained rather naturally in the CF theory, where the two states are related by particle hole symmetry in the fourth Λ level. States with excitations across one Λ level can also be identified in these spectra as forming a well-defined second band. Clearly, the structure of the low-energy states in the lowest LL is qualitatively distinct from that for H_3.

The above description of the UCF gives natural insight into many properties of the $5/2$ state.

Odd-even parity effect.—The presence of the topological exciton at odd N causes $O(1)$ oscillations in energy as a function of N, as found by Lu, Das Sarma, and Park [10], with the energy difference between odd and even N being equal to the minimum energy of the topological exciton.

Ordinary vs topological exciton.—A nontrivial outcome is that, in the large wave vector limit (namely, the large L limit in the spherical geometry), the UCF and ordinary neutral exciton have the same energy because, in this limit, the constituent quasiparticle and quasiholes are far separated and their energy does not depend on whether they reside in the same or different partitions. (The energy must be defined properly relative to the uniform ground state, which for odd N is to be obtained by interpolation.) For H_3 this has already been noted in exact diagonalization studies [4]. For H_C the finite size effects are stronger, but Fig. 5 convincingly demonstrates that the two exciton energies at large L converge with increasing N. At short distances (small L), on the other hand, the energies of the ordinary and topological excitons are *not* equal. Further, the

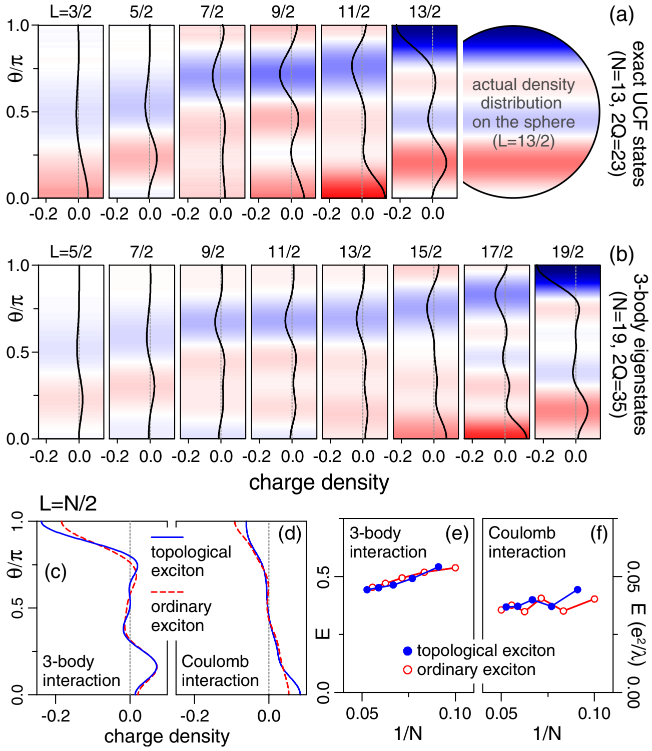

FIG. 5 (color online). Density profiles of (a) the unpaired CF state Ψ^{UCF} with $N = 13$ and (b) the 3-body eigenstate ($N = 19$) on a sphere as a function of the polar angle θ. The density, normalized to the filling factor, is measured relative to $1/2$. The different curves correspond to different L with $L_z = L$; $L = 3/2$ is not shown in (b) because it falls into the continuum. The distance between the constituent quasiparticle and quasihole increases with L; they are both near the north pole ($\theta = 0$) for $L = 3/2$ but at the opposite poles for $L = N/2$. (c) and (d) show comparisons of the density profiles of the topological exciton (with $N = 19$) and ordinary excitons ($N = 18$) at the largest $L = L_z = N/2$ for 3-body and 2nd LL Coulomb interactions; (e) and (f) show their energies as a function of N. The dimension of the $L_z = 1/2$ configuration space exceeds 107×10^6 for $N = 19$ at $2Q = 35$, the largest so far for which H_3 has been diagonalized.

ordinary CF exciton is known to display a complex dispersion, possibly with several "roton" minima, resulting from a complex interplay between the density profiles of the quasihole and the quasiparticle as a function of their separation [20,22–24]. Similar behavior can be expected for the topological exciton, the constituents of which also have complicated density profiles (Fig. 5), and, indeed, the dispersions in Fig. 2 or previous studies [4,5] do exhibit minima. The minimum energy of the UCF will thus be lower than its large L limit, as indicated by numerical studies [5]. The location of the minimum depends on the form of the interaction and will, in general, occur at different L for H_3 and H_C [5]. As noted in Ref. [4], observation of the topological exciton will require a probe that changes N by one unit; standard light scattering, which does not alter N, will excite the *ordinary* exciton for states with even or odd N, which differ only by a single localized topological exciton.

Fusion rules.—Using the standard terminology of the Ising conformal field theory [7], we identify the UCF by ψ and the vortex (a quasiparticle or a quasihole in one partition) by σ. The relation $\sigma \times \sigma = 1 + \psi$ indicates that a quasihole and a quasiparticle can be combined to produce two kinds of excitations, by placing them in the same or different partitions. The former, labeled "1," produces an ordinary exciton all of whose quantum numbers are zero, whereas the latter, labeled "ψ," produces a UCF. The relation $\sigma \times \psi = \sigma$ captures the reaction in which the addition of a quasiparticle or a quasihole to the UCF state annihilates half of the topological exciton to leave a single quasiparticle or quasihole. Finally, $\psi \times \psi = 1$ encapsulates the fact that two UCFs make two ordinary excitons. In all cases above, we have considered the lowest energy outcomes only. The fusion relations in the presence of several quasiparticles or quasiholes can similarly be derived.

Zero modes.—A nontrivial prediction of the $p_x \pm ip_y$ pairing scenario is the existence of a degenerate subspace of states for quasiparticles or quasiholes that differ in fermion number. Möller, Wójs, and Cooper [4] have shown that the average energies of a system with two quasiholes or quasiparticles are very close, modulo finite size uncertainties, with and without a UCF (i.e., for even or odd N). To see what insight the present work contributes, consider a state at even N with $2n$ quasiholes, n in each partition. Adding a UCF produces an "imbalanced" system with $n + 1$ and $n - 1$ quasiholes in the two partitions, because the quasiparticle of the topological exciton annihilates one of the quasiholes. This allows the immediate conclusion that, *provided* the quasiholes are far apart, the energy both before and after is simply $2n$ times the self-energy of an isolated quasihole. The same holds for a collection of quasiparticles.

One may question if these conclusions, which rely on the validity of Ψ^{UCF}, apply to the solutions of H_C. In this context, it is important to note that the above analysis does not depend on the details of the wave functions but only on the structure of the theory for the quasiparticles, quasiholes, and the UCF. To the extent that this structure continues adiabatically to H_C, the conclusions should carry over and should also be robust to weak corrections arising from finite width and Landau level mixing.

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