Three-body correlations and finite-size effects in Moore-Read states on a sphere

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(Received 5 March 2004; revised manuscript received 13 September 2004; published 19 January 2005)

Two- and three-body correlations in partially filled degenerate fermion shells are studied numerically for various interactions between the particles. Three distinct correlation regimes are defined, depending on the superharmonic, subharmonic, or harmonic forms of Haldane pair pseudopotential at short range. The harmonic form applies to electrons in the first excited Landau level (LL_1). Their correlations near half filling are confirmed to have a simple three-body form characteristic of the Moore-Read (MR) Pfaffian state, consisting of the maximum avoidance of the triplet state with the smallest relative angular momentum. To study MR correlations quantitatively, three-body amplitudes and pseudopotentials are introduced and calculated. Exact sum rules derived for the amplitudes imply that MR correlations depend on the actual (pair) interaction largely through the short-range behavior of the corresponding three-body interaction. In the calculated spectra of a model three-body repulsion, higher-energy bands are identified in addition to (previously known) MR ground state and its elementary charged (quasiparticle) and neutral (pair-breaking) excitations. The pair-breaker dispersion curve is determined, and the quasiparticles are correctly described by a generalization of the composite fermion model appropriate for Halperin *p*-type electron pairing with Laughlin correlations between the pairs. The known problem of the exact few-electron ground states having small overlaps with MR trial states on a sphere is also resolved by showing that the short-range three-body pseudopotential coefficients are more sensitive to the surface curvature than the two-body ones (leading to a slow convergence of the overlaps as a function of the electron number). Hence, the MR state and its excitations are believed to be a more accurate description of the experimental $\nu=\frac{5}{2}$ quantum Hall states than could be inferred from previous, small-size calculations.

DOI: 10.1103/PhysRevB.71.045324 PACS number(s): 73.43.−f, 71.10.Pm

I. INTRODUCTION

The fractional quantum Hall (FQH) effect[1,2] is a many-body phenomenon consisting of the quantization of Hall conductance and the simultaneous vanishing of longitudinal resistance of a high-mobility quasi-two-dimensional electron gas at a strong magnetic field B and low density ϱ, corresponding to certain universal fractional values of the Landau level (LL) filling factor $\nu=2\pi\varrho\lambda^2$ (where $\lambda=\sqrt{hc/eB}$ is the magnetic length). This macroscopic effect is a consequence of the formation of incompressible liquid ground states (GS's) with quasiparticle (QP) excitations.[2] It depends on correlations in partially filled degenerate LL's, entirely determined by a Haldane pseudopotential[3] defined as the pair interaction energy V_2 as function of relative pair angular momentum $\mathcal{R}_2$.

The most prominent FQH states in LL_0 (lowest LL) are given by Jain sequence[4] of filled composite fermion[5,6] (CF) levels. In Haldane hierarchy,[7–11] they result from Laughlin correlations[12–14] (between electrons or QP's) induced by pseudopotentials that are superharmonic[15–17] at short range. However, the FQH states with different, non-Laughlin correlations occur as well. E.g., pairing in a half-filled LL_1 (first excited LL) is firmly established in the $\nu=\frac{5}{2}$ state,[18–21] while correlations between CF's in their CF-LL_1 responsible for the FQH effect[22,23] at $\nu=\frac{3}{8}$ or $\frac{4}{11}$ are not yet completely understood.

The lack of superharmonic behavior of the pseudopotential at short range together with the occurrence of clearly non-Laughlin half-filled FQH states suggests pairing in both LL_1 and CF-LL_1. Proposed trial states include Halperin[12] and Haldane-Rezayi[24] states with Laughlin correlations between spin-triplet and -singlet pairs, respectively, and the Moore-Read[25,26] Pfaffian state that can be defined as a zero-energy ground state of a short-range three-body repulsion.[27] These pair states have all been studied in great detail[28–34] because of their anticipated exotic properties, such as non-Abelian QP statistics[25] or existence of pair-breaking (PB) neutral fermion excitations.[27,35] However, choosing the correct one for specific real FQH systems is somewhat problematic. In the following we concentrate on the half-filled LL_1, and pairing in CF-LL_1 is addressed elsewhere.[36]

The trouble with Halperin state[12] (in which the electrons form tightly-bound pairs, and these pairs form an even-denominator Laughlin state, as appropriate for bosons) is that because the relative angular momentum of the constituent pairs is not a conserved quantity, it is more of an intuitive concept for the correlations than a well-defined trial wave function obeying all required symmetries. For example, description of the pair-pair interaction by an effective pseudopotential is not rigorous,[36] and the harmonic criterion[16,17] that would relate the occurrence of Laughlin pair-pair correlations with an electron pseudopotential is not exact. Consequently, it has not been clear what model interaction induces such correlations (and ground state). In fact, it has been (erroneously) assumed[27] that this paired state results for pseudopotentials attractive at short range rather than harmonically repulsive as in LL_1, which would suggest that it is not an adequate trial state for the $\nu=\frac{5}{2}$ FQH effect.

The Moore-Read wave function on the other hand is well defined.[25–27] However, it only occurs for interactions with very particular short-range behavior, while the pseudopotentials in realistic experimental systems depend on sample parameters such as the layer width w, magnitude, tilt of the magnetic field, etc. Moreover, finite-size calculations indicate that realistic Coulomb pseudopotentials are too weak at short range (by up to $\sim 10\%$ for $w=0$) to induce a Moore-Read ground state.[26,37] This would seem to imply that the Moore-Read state does not describe the $\nu=\frac{5}{2}$ FQH state quite as accurately as a Laughlin state describes the actual $\nu=\frac{1}{3}$ ground states. The occurrence of the $\nu=\frac{5}{2}$ FQH effect could still be attributed to the observation that the calculated excitation gaps are much less sensitive to the details of the pseudopotential than the wave functions. However, poor accuracy of the Moore-Read wave function puts doubt on the occurrence of those of its properties in realistic $\nu=\frac{5}{2}$ systems that depend more critically on the correlations. As these properties (including non-Abelian QP's) are so much more fascinating than plain incompressibility, the question of whether they indeed remain only an unrealized theoretical concept is quite significant. Theoretical insight is especially valuable in this problem because of the difficulty with direct experimental evidence.[38]

II. OUTLINE

The main subject of this paper is the analysis of two- and three-body correlations in half-filled LL's with different pair interactions, especially in a half-filled LL_1. The most important results are (i) application of three-body pseudopotentials and amplitudes to study three-body correlations, (ii) direct demonstration of "Moore-Read correlations" consisting of the tendency to maximally avoid the "compact" triplet state with the smallest relative angular momentum in the low-energy many-body states near half filling, (iii) discovery of the equivalence of Halperin pairing and the Moore-Read Pfaffian state, and (iv) resolution of the problem of small overlaps between exact few-electron ground states and Moore-Read trial states on a sphere.

Because a number of minor original conclusions are intermixed with those known or anticipated previously (that nevertheless are often invoked to make the paper more self-contained), below we give a more detailed list of new results presented in the following sections. In Sec. III we compare Haldane pair pseudopotentials $V_2(\mathcal{R}_2)$ and pair amplitudes $\mathcal{G}_2(\mathcal{R}_2)$ calculated at $\frac{1}{2}$ and $\frac{1}{3}$ filling of LL_0, LL_1, and CF-LL_1.

In Sec. IV we extend Haldane's concept and introduce *triplet* pseudopotentials V_3 and amplitudes $\mathcal{G}_3$, defined as functions of triplet relative angular momentum $\mathcal{R}_3$. A sum rule is derived for the amplitudes which implies that only anharmonic[39] contributions to V_3 induce triplet correlations. This result justifies the use of a short-range three-body repulsion W that is nonzero only for the minimum allowed value of $\mathcal{R}_3=3$ to model actual (Coulomb) interaction in LL_1. We also demonstrate that when V_2 is nearly harmonic at short range (as in LL_1), then V_3 is slightly superharmonic at small $\mathcal{R}_3$, and simple "Moore-Read" triplet correlations result at half filling, consisting of a complete avoidance of $\mathcal{R}_3=3$. Therefore, rather than *assuming* that the Moore-Read wave function describes the FQH $\nu=\frac{5}{2}$ state, we demonstrate directly to what extent its defining property is realized in this state. We also find that $\mathcal{R}_3=3$ is not avoided when V_2 is subharmonic (as in CF-LL_1), proving that the origin of incompressibility at $\nu=\frac{3}{8}$ is different.

In Sec. V we analyze the energy spectra of W (some known previously for smaller systems) and confirm the occurence of the zero-energy Moore-Read state and its elementary charged (QP) and neutral (PB) excitations. A continuous PB dispersion is determined, and the QP's are correctly described by a generalization of the composite fermion model appropriate for Halperin p-type electron pairing with Laughlin correlations between the pairs. In particular, higher bands in the spectra of W are identified as containing additional quasielectron-quasihole (QE-QH) pairs and explained using Halperin picture.[12]

In Sec. VI we calculate (on a sphere) overlaps of different trial states with the corresponding few-particle eigenstates of various pair interactions. While the overlaps of the Moore-Read state with electron ground states in LL_1 have been published previously, here we also calculate them for all elementary excitations (to verify that Moore-Read correlations are a valid description also beyond the ground state) and show their dependence on the short-range harmonicity of pseudopotential. The discussion of finite-size errors and of resolution of the known problem of small overlaps on a sphere by explaining their slow convergence with the electron number are important because the "numerical experiments" in small systems are the most reliable tool for identifying the $\nu=\frac{5}{2}$ FQH state as a Moore-Read state.

III. TWO-BODY CORRELATIONS

A. Haldane pair pseudopotential

Within a degenerate LL, the many-body Hamiltonian only contains the interaction term, which is completely determined by the discrete (Haldane) pseudopotential $V_2(\mathcal{R}_2)$ defined as pair interaction energy V_2 as a function of relative pair angular momentum $\mathcal{R}_2$. For identical fermions/bosons, $\mathcal{R}_2$ takes on odd/even integer values, respectively, and the larger $\mathcal{R}_2$ corresponds to a larger average pair separation $\sqrt{\langle r^2\rangle}$. On a sphere, $\mathcal{R}_2=2l-L_2$, where l is the single-particle angular momentum of the shell (LL), and L_2 is the total pair angular momentum. (We use the following standard notation for Haldane[7] spherical geometry: $l=Q+n$ for the nth LL, $2Q=4\pi R^2 B/\phi_0$ is the magnetic monopole strength $\phi_0=hc/e$ is the flux quantum, R is the sphere radius, and $\lambda=R/\sqrt{Q}$ is the magnetic length.) Importantly, $V_2(\mathcal{R}_2)$ combines information about both interaction potential $V(r)$ and the single-particle wave functions allowed within the Hilbert space restricted to a LL. The pseudopotentials obtained for the electrons in LL_0 and LL_1, and for Laughlin QE's in CF-LL_1 are shown in Fig. 1. The result for CF-LL_1 is obtained[11,40] from the N-electron energy spectra in LL_0, at the value of $2l=3N-5$ corresponding to a pair of QE's in the Laughlin $\nu=\frac{1}{3}$ state (within the lowest bands of such spectra,

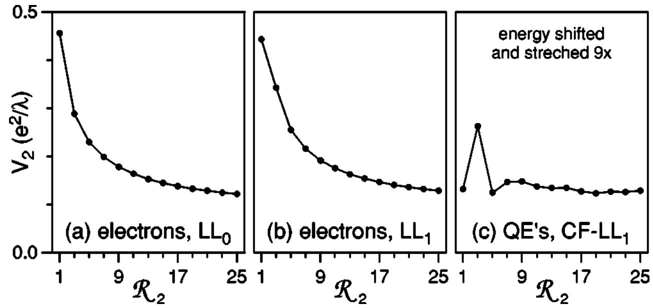

FIG. 1. Pair interaction pseudopotentials (pair interaction energy V_2 vs relative pair angular momentum $\mathcal{R}_2$) for electrons in the lowest (a) and first excited LL (b), and for QE's of the Laughlin $\nu=\frac{1}{3}$ state (c). The values of V_2 in frame (c) were calculated by Lee *et al.* (Ref. 40) and are only known up to a constant. λ is the magnetic length.

the dependence of energy on angular momentum is, up to a constant, the QE-QE pseudopotential).

B. Pair amplitudes

The pair correlations induced by a specific $V_2(\mathcal{R}_2)$ are conveniently described by a discrete pair amplitude function $\mathcal{G}_2(\mathcal{R}_2)$, defined[3] as the number of pairs $\mathcal{N}_2$ with a given $\mathcal{R}_2$ divided by the total pair number

$$\mathcal{G}_2(\mathcal{R}_2) = \binom{N}{2}^{-1} \mathcal{N}_2(\mathcal{R}_2). \tag{1}$$

It immediately follows from the expression for the total interaction energy of an N-body state

$$E = \binom{N}{2} \sum_{\mathcal{R}_2} \mathcal{G}_2(\mathcal{R}_2) V_2(\mathcal{R}_2) \tag{2}$$

that the low-energy many-body states generally have a large/small amplitude at those values of $\mathcal{R}_2$ corresponding to small/large repulsion $V_2(\mathcal{R}_2)$. In Fig. 2 we compare the pair amplitudes obtained in Haldane spherical geometry for $N=12$ and 14 particles confined in angular momentum shells with degeneracy $g=2l+1$ corresponding to the filling factors $\nu\sim\frac{1}{3}$ and $\frac{1}{2}$ and interacting through the pseudopotentials of Fig. 1. Although for each system (V_2,N,g) we only show the data for the lowest $L=0$ state, virtually identical $\mathcal{G}_2(\mathcal{R}_2)$ functions are obtained for all low-energy states of each system.

The chosen values of $2l$ and N correspond to three different sequences of finite-size spherical systems known to represent the following FQH states observed experimentally on a plane. The $2l=2N-3$ sequence describes the paired Moore-Read[25] $\nu=\frac{1}{2}$ state in LL_1 (corresponding to the total electron filling factor $\nu=\frac{5}{2}$) and the $\nu=\frac{1}{2}$ state of QE's in CF-LL_1 identified numerically[36] for $N=6$, 10, and 14, and corresponding to the FQH effect[22] at $\nu=\frac{3}{8}$. The $2l=3N-7$ sequence describes the (not well understood) $\nu=\frac{1}{3}$ states in both LL_1 (Ref. 17) and CF-LL_1,[36] corresponding to the

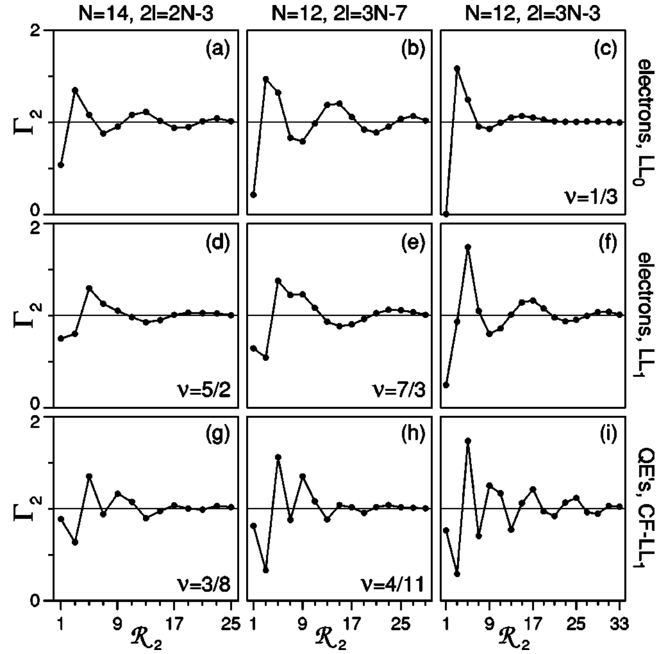

FIG. 2. Pair-correlation functions (pair amplitude Γ_2 vs relative pair angular momentum $\mathcal{R}_2$) calculated on a sphere for the lowest $L=0$ states of N particles interacting through pseudopotentials shown in Fig. 1, at values of $2l$ corresponding to different FQH states at filling factors $\nu=\frac{1}{2}$ and $\frac{1}{3}$.

$\nu=\frac{7}{3}$ state[18–21] and $\nu=\frac{4}{11}$ state,[22,23] respectively. Finally, the $2l=3N-3$ sequence describes the Laughlin[2] $\nu=\frac{1}{3}$ state in LL_0.

The pair amplitude calculated for a completely filled shell (the $\nu=1$ state) with a given $2l$ is a decreasing straight line

$$\mathcal{G}_2^{\text{full}}(\mathcal{R}_2) = \frac{4l+1-2\mathcal{R}_2}{l(2l+1)}, \tag{3}$$

which is a finite-size edge effect. In the $2l\to\infty$ limit corresponding to an infinite plane, $\mathcal{N}_2^{\text{full}}(\mathcal{R}_2)=N$ and the ratio $\mathcal{N}_2^{\text{full}}(\mathcal{R}_2)/N \equiv \Gamma^{\text{full}}(\mathcal{R}_2)=1$ is the appropriately renormalized pair amplitude in this geometry.

The overall linear decrease of $\mathcal{G}_2(\mathcal{R}_2)$ appears also at $\nu<1$, and it should be ignored in the analysis of correlations. Therefore, in Fig. 2 we actually plot

$$\Gamma_2(\mathcal{R}_2) = 1 + \frac{\mathcal{G}_2(\mathcal{R}_2) - \mathcal{G}_2^{\text{full}}(\mathcal{R}_2)}{\mathcal{G}_2^{\text{full}}(1)}, \tag{4}$$

in which the linear decrease is eliminated and the scaling appropriate for an infinite plane is used, to ensure that $\Gamma_2(1)\propto\mathcal{G}_2(1)$, that $\Gamma_2(\mathcal{R}_2)=1$ for finite-size $\nu=1$ states, and that $\Gamma_2(\mathcal{R}_2)$ converges to the pair-correlation function on the plane when N is increased.

In all frames, Γ_2 is significantly different from 1 only at small $\mathcal{R}_2$, and the oscillations around this value quickly decay beyond $\mathcal{R}_2\sim 7$. This can be interpreted as a short correlation range ξ in all studied systems and it justifies the use of finite-size calculations (requiring that $\xi \ll R$).

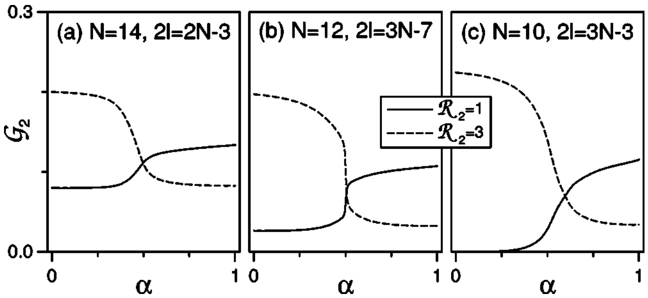

FIG. 3. Dependence of pair amplitudes $\mathcal{G}_2$ on parameter α of pair interaction U_α defined by Eq. (5), calculated on a sphere for the lowest $L=0$ states of N-particle systems representing the same FQH states as used in Fig. 2.

Clearly, three different interactions result in quite different correlations. In LL_0 (a)–(c), the dominant tendency is the avoidance of $\mathcal{R}_2=1$ (Laughlin correlations) at a cost of having a large number of pairs with $\mathcal{R}_2=3$. Around half filling of LL_1 (d),(e), the numbers of pairs with $\mathcal{R}_2=1$ and 3 are about equal and both small. Finally, in a partially filled CF-LL_1 (g)–(i), the $\mathcal{R}_2=3$ pair state is maximally avoided.

C. Model interaction and pair-correlation regimes

The fact that $\Gamma_2 \approx 1$ at long range also explains why the low-energy wave functions are virtually insensitive to the exact form of $V_2(\mathcal{R}_2)$ beyond a few leading parameters at $\mathcal{R}_2=1,3,\ldots$. Furthermore, due to the sum rules obeyed by pair amplitudes,[16,17,41] the harmonic pseudopotentials $V_2^H(\mathcal{R}_2)=c_0-c_1\mathcal{R}_2$ (with constant c_0 and c_1) induce no correlations, and only the anharmonic contributions to $V_2(\mathcal{R}_2)$ at small $\mathcal{R}_2$ (short range) affect the pair-correlation functions. Indeed, simple model pseudopotentials with only two nonvanishing leading parameters are known[17,27,36] to accurately reproduce correlations shown in Fig. 2. Let us define such $U_\alpha(\mathcal{R}_2)$ with

$$U_\alpha(1) = 1 - \alpha,$$

$$U_\alpha(3) = \alpha/2. \tag{5}$$

U_0 and U_1 are the two extremal pseudopotentials with only one anharmonic term. $U_{1/2}$ is harmonic through $\mathcal{R}_2=1$, 3, and 5, and thus it favors equally the avoidance of both $\mathcal{R}_2=1$ and 3 pairs (any distribution of the pair amplitude between the $\mathcal{R}_2=1$, 3, and 5 states that satisfies the sum rules yields the same total energy E).

In Fig. 3 we plot $\mathcal{G}_2(1)$ and $\mathcal{G}_2(3)$ as a function of α for the lowest $L=0$ state in three finite-size systems representing the same series of FQH states as used in Fig. 2. The correlations in a partially filled LL_0 (Laughlin correlations), LL_1, and CF-LL_1 are well reproduced by U_α with $\alpha \approx 0$, $\frac{1}{2}$, and 1, respectively. The correlations at $\alpha=0$ (i.e., in LL_0) and at $\alpha=1$ (i.e., in CF-LL_1) can easily be expressed in terms of pair amplitudes. With $U(1)$ or $U(3)$ being the only nonvanishing (and positive) coefficient, it follows from Eq. (2) that the low-energy states must have the minimum allowed (within the available Hilbert space) $\mathcal{G}_2(1)$ or $\mathcal{G}_2(3)$, respectively.

It is well known that for Laughlin correlations, because of the simple form of single-particle wave functions in LL_0, the complete avoidance of $\mathcal{R}_2=1$ pairs (possible at $\nu \leq \frac{1}{3}$) appears in form of a Jastrow factor in the Laughlin wave function. It justifies the mean-field CF picture that essentially attributes reduction of the many-body degeneracy caused by a $\mathcal{R}_2=1$ hardcore (or "correlation hole") to an effective, reduced magnetic field.

For QE's, the tendency to have small $\mathcal{G}_2(3)$ and, consequently, significant $\mathcal{G}_2(1)$ (compared to a Laughlin-correlated state at the same ν) has been interpreted[36] as pairing or clustering. However, the numerical studies are not conclusive about how the clusters correlate with one another, and the question of the origin of the excitation gap observed at $\nu=\frac{3}{8}$ or $\frac{4}{11}$ remains open.

In a partially filled LL_1 the situation is more complicated. Because $V_2(\mathcal{R}_2)$ is nearly harmonic at short range $(\alpha \sim \frac{1}{2})$, the energy is nearly independent of the relative occupation of the $\mathcal{R}_2=1$ and 3 pair states. Therefore, the correlations cannot be easily expressed in terms of pair amplitudes [although the linear combination of $\mathcal{G}_2(1)$ and $\mathcal{G}_2(3)$ equal to the total energy E is obviously minimized at its corresponding value of α]. However, it turns out that it is the short-range three-body correlations that determine the low-energy states in this regime. Soon after its introduction, the half-filled Moore-Read state was shown[27] to be an exact zero-energy eigenstate of a model short-range three-body repulsion, and the spectra of this interaction were later studied in detail.[28,31] Below we analyze the three-body correlations directly, by the calculation of an appropriate correlation function.

IV. THREE-BODY CORRELATIONS

A. Three-body pseudopotential

In analogy to the avoidance of the strongly repulsive pair states, the three-body states with sufficiently high energy (compared to the rest of the three-body spectrum) will also be avoided in the low-energy many-body states. For the pairs, the eigenstates are uniquely labeled by $\mathcal{R}_2$, and the criterion for the avoidance of a specific $\mathcal{R}_2$ is[16,17] that it corresponds to the dominant positive anharmonic term of $V_2(\mathcal{R}_2)$. The three-body states are also labeled by the relative (with respect to the center of mass) angular momentum $\mathcal{R}_3$. The allowed values are $\mathcal{R}_3=3$ or $\mathcal{R}_3 \geq 5$, and larger $\mathcal{R}_3$ means larger expectation value of the area spanned by the three particles.[42] On a sphere, $\mathcal{R}_3=3l-L_3$, where L_3 is the total triplet angular momentum.

Since no degeneracies appear in the $V_3(\mathcal{R}_3)$ energy spectrum for $\mathcal{R}_3<9$, its low-$\mathcal{R}_3$ part can be considered a three-body pseudopotential analogous to $V_2(\mathcal{R}_2)$. The three-body pseudopotentials $V_3(\mathcal{R}_3)$ obtained for different pair pseudopotentials $V_2(\mathcal{R}_2)$ of Fig. 1 are shown in the upper frames of Fig. 4. The nonmonotonic behavior of $V_3(\mathcal{R}_3)$ in frame (c) most likely precludes the tendency to avoid the $\mathcal{R}_3=3$ triplet state in QE systems. On the other hand, it seems plausible that the monotonic character of $V_3(\mathcal{R}_3)$ in frame (b) might lead to the avoidance of the same $\mathcal{R}_3=3$ triplet state in a partially filled LL_1.

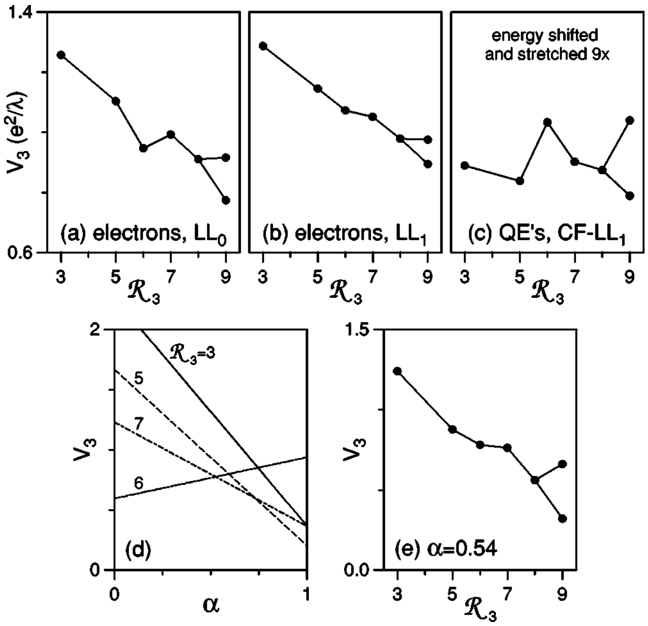

FIG. 4. (a)–(c) Triplet interaction pseudopotentials (triplet interaction energy V_3 vs relative triplet angular momentum $\mathcal{R}_3$) for pair pseudopotentials shown in Fig. 1. λ is the magnetic length. (d) Dependence of coefficients V_3 on parameter α of pair interaction U_α defined by Eq. (5). (e) Pseudopotential $V_3(R_3)$ calculated for pair interaction $U_{0.54}$.

The dependence of $V_3(\mathcal{R}_3)$ on $V_2(\mathcal{R}_2)$ can be captured by plotting the leading V_3 coefficients as a function of parameter α of the model pair pseudopotential U_α, as shown in frame (d). For $\mathcal{R}_3 < 9$ the triplet wave functions are fixed and so are their $\mathcal{G}_2$ amplitudes, and hence the dependences $V_3(\alpha)$ are all linear. Only around $\alpha \sim \frac{1}{2}$ is the $V_3(\mathcal{R}_3)$ function superlinear for small $\mathcal{R}_3$, as shown on an example for $\alpha=0.54$ in frame (e).

B. Three-body amplitudes

In order to test the hypothesis of the avoidance of the $\mathcal{R}_3=3$ triplet eigenstate in partially filled LL_1, we introduce "triplet amplitude" $\mathcal{G}_3(\mathcal{R}_3)$. It is defined in analogy to Haldane pair amplitude,[3] as an expectation value of the operator $\hat{P}_{ijk}(\mathcal{R}_3,\beta_3)$ projecting a many-body state Ψ onto the subspace in which the three particles ijk are in an eigenstate $|\mathcal{R}_3,\beta_3\rangle$ (here, β_3 is an additional index to distinguish degenerate multiplets at the same $\mathcal{R}_3$; it can be omitted for $\mathcal{R}_3 < 9$). The interaction Hamiltonian written in a three-body form using $\hat{\mathcal{P}}_{ijk}$ reads

$$\hat{\mathcal{H}} = \sum_{i<j<k} \sum_{\mathcal{R}_3,\beta_3} \hat{\mathcal{P}}_{ijk}(\mathcal{R}_3,\beta_3)\, V(\mathcal{R}_3,\beta_3). \tag{6}$$

The triplet amplitude is

$$\mathcal{G}_3(\mathcal{R}_3,\beta_3) = \binom{N}{3}^{-1} \langle\Psi| \sum_{i<j<k} \hat{\mathcal{P}}_{ijk}(\mathcal{R}_3,\beta_3)|\Psi\rangle, \tag{7}$$

which for a totally antisymmetric Ψ is equivalent to

$$\mathcal{G}_3(\mathcal{R}_3,\beta_3) = \langle\Psi|\hat{\mathcal{P}}_{123}(\mathcal{R}_3,\beta_3)|\Psi\rangle. \tag{8}$$

Triplet amplitudes defined in this way are normalized to

$$\sum_{\mathcal{R}_3,\beta_3} \mathcal{G}_3(\mathcal{R}_3,\beta_3) = 1, \tag{9}$$

so that they measure the fraction of all triplets being in a given eigenstate

$$\mathcal{G}_3(\mathcal{R}_3,\beta_3) = \binom{N}{3}^{-1} \mathcal{N}_3(\mathcal{R}_3,\beta_3). \tag{10}$$

The energy of Ψ is expressed as

$$E = \binom{N}{3} \sum_{\mathcal{R}_3,\beta_3} \mathcal{G}_3(\mathcal{R}_3,\beta_3) V_3(\mathcal{R}_3,\beta_3). \tag{11}$$

On a sphere, triplet amplitudes are connected with the third-order parentage coefficients[43] $G_3(L_3,\beta_3;L_3',\beta_3')$, i.e., the expansion coefficients of a totally antisymmetric state Ψ in a basis of product states in which particles (1,2,3) and $(4,5,\ldots,N)$ are in the 3- and $(N-3)$-body eigenstates $|L_3,\beta_3\rangle$ and $|L_3',\beta_3'\rangle$, respectively,

$$\mathcal{G}_3(L_3,\beta_3) = \sum_{L_3',\beta_3'} |G_3(L_3,\beta_3;L_3',\beta_3')|^2. \tag{12}$$

Note that to obey standard notation for parentage coefficients, in the above equation we use total angular momentum L_3 instead of the relative one, $\mathcal{R}_3 = 3l - L_3$, to label triplet states. Also, we omit index Ψ in $\mathcal{G}_3$ and G_3.

The operator identity[41]

$$\hat{L}^2 + N(N-2)\hat{l}^2 = \sum_{i<j} \hat{L}_{ij}^2 \tag{13}$$

connects the total N-body angular momentum (L) with the single-particle and pair angular momenta l and L_{ij}. We used it earlier to show that harmonic pair pseudopotentials cause no correlations. Here, we generalize it to the form

$$\hat{L}^2 + \frac{N(N-K)}{K-1}\hat{l}^2 = \binom{N-2}{K-2}^{-1} \sum_{i_1<\cdots<i_K} \hat{L}_{i_1\cdots i_K}^2. \tag{14}$$

By taking the expectation values of both sides of the above equation in the (totally antisymmetric) state Ψ and using the expansion of Ψ in terms of the Kth-order parentage coefficients we obtain

$$L(L+1) + \frac{N(N-K)}{K-1} l(l+1) = \frac{N(N-1)}{K(K-1)} \times \sum_{L_K,\beta_K} \mathcal{G}_K(L_K,\beta_K) L_K(L_K+1), \tag{15}$$

an additional (besides normalization) sum rule obeyed by the amplitudes $\mathcal{G}_K$.

Just as for the specific $K=2$ case discussed earlier,[41] the above sum rule (15) together with an appropriate version of Eq. (2) or (11) immediately implies that if the K-body interaction pseudopotential V_K is linear in $L_K(L_K+1)$, all N-body

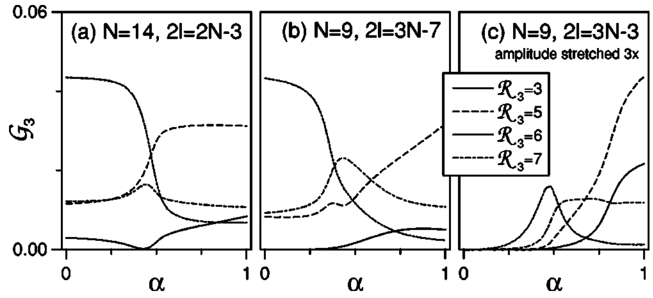

FIG. 5. Dependence of triplet amplitudes $\mathcal{G}_3$ on parameter α of pair interaction U_α defined by Eq. (5), calculated on a sphere for the lowest $L=0$ states of N-particle systems representing the same FQH states as used in Figs. 2 and 3.

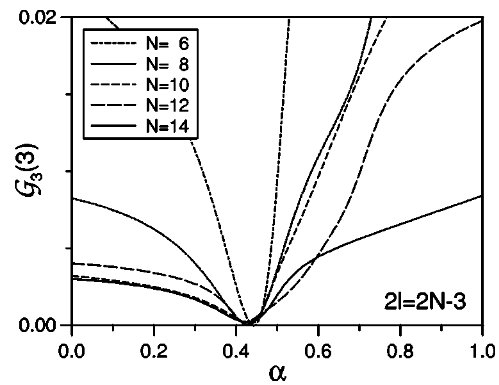

FIG. 6. The $\mathcal{G}_3(3)$ vs α curve for $\nu=\frac{1}{2}$ shown in Fig. 5(a), magnified and replotted for different particle numbers N.

multiplets with the same L are degenerate. In the limit of infinite LL degeneracy $g=2l+1$ corresponding to an infinite sphere radius (vanishing curvature), i.e., to the planar geometry, the linearity in $L_K(L_K+1)$ translates into the linearity in $\mathcal{R}_K=Kl-L_K$, and it turns out that the linear part of $V_K(\mathcal{R}_K)$ causes no correlations.

For $K=3$, this is precisely what justifies the use[28,31] of a model short-range three-body repulsion as simple as $W(\mathcal{R}_3)$ of Eq. (17) to model actual (Coulomb) interactions in LL_1. Equation (15) shows that W does not have to be understood as a three-body pseudopotential of an (unrealistic) contact three-body repulsion[27] $V_{i;jk}=\Sigma_{ijk}\delta(z_{ij})\delta(z_{ik})$, but merely as a dominant anharmonic contribution to V_3 at small $\mathcal{R}_3$.

C. Three-body correlation hole

Let us now turn back to the numerical results. In Fig. 5 we plot the dependence of the leading $\mathcal{G}_3(\mathcal{R}_3)$ coefficients on α, calculated in the lowest $L=0$ state of three different systems belonging to the same sequences of finite-size FQH states as used earlier in Figs. 2 and 3. Clearly, all triplet amplitudes significantly depend on α, but we especially want to point out the following three features for $\mathcal{R}_3=3$: (i) the tendency to avoid $\mathcal{R}_2=1$ pairs at $\alpha\sim 0$ is *not* synonymous with the avoidance of $\mathcal{R}_3=3$ triplets at $\nu=\frac{1}{2}$, (ii) $\mathcal{G}_3(3)$ *vanishes* for $\alpha\approx\frac{1}{2}$ at $\nu=\frac{1}{2}$, (iii) $\mathcal{G}_3(3)$ *increases* when α increases beyond $\frac{1}{2}$ in all frames.

Before we concentrate on the Moore-Read state, let us note that observation (iii) implies that electron correlations at $\nu=\frac{5}{2}$ are distinctly different from QE correlations at $\nu=\frac{3}{8}$, even though these two (incompressible) states belong to the same "universality class"[27] (one smoothly evolves into the other without loss of incompressibility under a continuous transformation of the pair pseudopotential). This confirms the suspicion based on the form of triplet pseudopotential $V_3(\mathcal{R}_3)$ of Fig. 4(c) that (against an earlier assumption[27]) Halperin paired state[12] is not an adequate description for systems with subharmonic pseudopotentials at short range. In particular (against our earlier expectation[44] but in agreement with our later numerical results[36]) such model appears inappropriate for the QE's in CF-LL_1 at $\nu=\frac{1}{2}$ or $\frac{1}{3}$, corresponding to the FQH states at $\nu=\frac{3}{8}$ and $\frac{4}{11}$. Instead of Halperin pairing, grouping of pairs into larger clusters seems to occur for the QE's, although we are not able to define their correlations more specifically.

Let us now discuss observations (i)–(iii) in more detail. In Fig. 6 we plot $\mathcal{G}_3(3)$ as a function of α for $N=6$ to 14 (only even values, because the Moore-Read state at $2l=2N-3$ is a paired state). For each N, $\mathcal{G}_3(3)$ drops to essentially a zero at $\alpha_0\approx\frac{1}{2}$. This result is consistent with the calculations of overlaps of the exact ground states of modified Coulomb interaction with the exact Moore-Read trial state.[26,37]

Note that although the avoidance of $\mathcal{R}_3=3$ triplets at $\nu=\frac{1}{2}$ can be extracted from the Moore-Read wave function, we demonstrate that it really occurs in a system with pair interactions. Therefore, rather than *assuming* that the Moore-Read wave function correctly describes the $\nu=\frac{1}{2}$ state in LL_1, we demonstrate directly to what extent its defining property is realized in this system. This result has not been presented so quantitatively before.

It is difficult to reliably extrapolate the values of α_0 obtained from Fig. 6 to an infinite (planar) system. However, we notice the following connection with Fig. 4(d) that depends on $2l$ much more regularly. The pair amplitudes $\mathcal{G}_2=[\mathcal{G}_2(1),\mathcal{G}_2(3),\ldots]$ of the $\mathcal{R}_3<9$ triplets can be calculated. On a sphere, they slightly depend on $2l$ (on curvature), but the values appropriate for a plane (the $g\to\infty$ limit) are $\left[\frac{3}{4},\frac{1}{4}\right]$, $\left[\frac{9}{16},\frac{1}{8},\frac{5}{16}\right]$, and $\left[\frac{3}{16},\frac{5}{8},\frac{3}{16}\right]$ for $\mathcal{R}_3=3$, 5, and 6, respectively. Using these values and Eq. (2) one can determine the range α over which $V_3(\mathcal{R}_3)$ is superlinear at short range. The requirement that $\frac{1}{2}[V_3(3)-V_3(5)]>V_3(5)-V_3(6)$ and $V_3(5)-V_3(6)>V_3(6)-V_3(7)$ limits α to a rather narrow window of approximately

$$0.5<\alpha+\frac{1}{4l}<0.58. \tag{16}$$

For a reason we do not completely understand [but that is connected with a neglected and complicated behavior of $V_3(\mathcal{R}_3)$ at $\mathcal{R}_3>7$], the value of α_0 in finite systems (see Fig. 6) is much closer to the lower limit of Eq. (16). Therefore, we expect that α_0 will follow this lower limit with increasing $2l$, and the value appropriate for a planar system should be even closer to $\frac{1}{2}$ than the finite-size results of Fig. 6. And

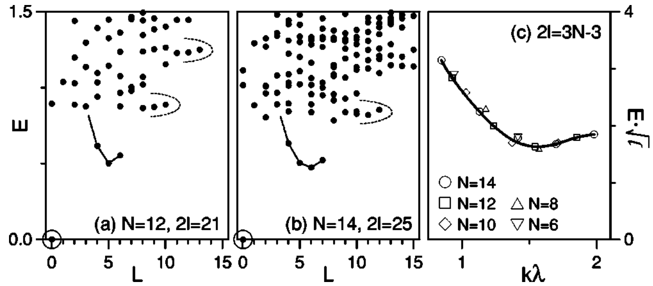

FIG. 7. (a), (b) Energy spectra (total interaction energy E vs total angular momentum L) calculated on a sphere for even numbers of particles N interacting through triplet pseudopotential defined by Eq. (17), at the values of $2l=2N-3$ corresponding to the $L=0$ Moore-Read ground state. (c) Energy dispersion (excitation energy E as a function of wavevector k) for the excited magnetoroton band. λ is the magnetic length. Similar results were first obtained by Read and Rezayi (Ref. 31).

since $U_{1/2}$ accurately models Coulomb interaction in LL_1, we conclude that the "$\mathcal{R}_3>3$" correlations must be an accurate description for experimental $\nu=\frac{5}{2}$ FQH state (even in narrow samples). This conclusion is quite different from an earlier discussion of finite-size numerical wave functions[26,37] which seemed to imply that a ~10% short-range enhancement of the Coulomb pseudopotential calculated for $w=0$ in LL_1 is needed to reach good overlap with the Moore-Read state.

V. ENERGY SPECTRA OF SHORT-RANGE THREE-BODY REPULSION

Knowing that what defines the Moore-Read state is that electrons in $\frac{1}{2}$-filled LL_1 completely avoid the $\mathcal{R}_3=3$ triplet state,[27] let us discuss the energy spectra of the model short-range three-body repulsion

$$W(\mathcal{R}_3)=\delta_{\mathcal{R}_3,3} \tag{17}$$

which induces precisely this type of correlations. Similar calculations for slightly smaller systems were earlier carried out by Wen[28] and by Read and Rezayi.[31] Our spectra in Figs. 7 and 8(a) are equivalent, shown here for completeness and to identify (ignored before) higher-energy bands, while spectra in Figs. 8(b), 8(c), and 9 are original. There is no disagreement between our numerics and previous work,[28,31] but we

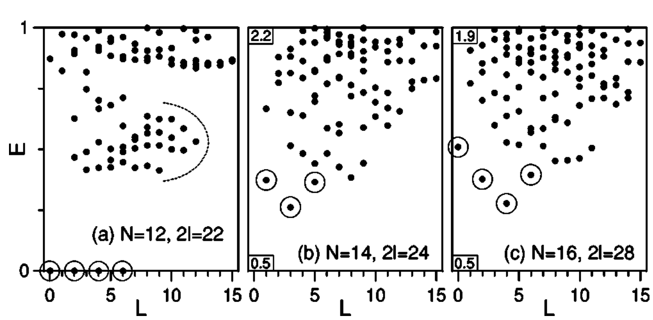

FIG. 8. The same as Fig. 7 but for $2l=2N-2$ (a) and $2l=2N-4$ (b), (c) corresponding to two QH's and two QE's in the Moore-Read state, respectively.

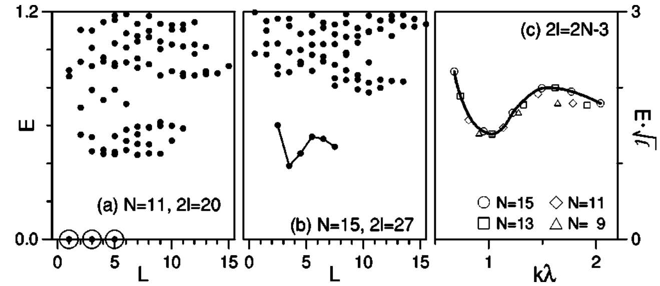

FIG. 9. The same as Figs. 7 and 8 but for odd particle numbers N and for $2l=2N-2$ (a) and $2l=2N-3$ (b). Energy dispersion (c) is for the pair-breaking band marked in frame (b). λ is the magnetic length.

present an interpretation using Halperin pairing picture.

The three-body interaction matrix elements needed for diagonalization in the configuration interaction (CI) basis are connected with the triplet spectrum $V_3(\mathcal{R}_3,\beta_3)$ through expansion parameters $C_B^A\equiv\langle A|B\rangle$ analogous to the pair Clebsch-Gordan coefficients

$$\langle m_1,m_2,m_3|V_3|m_4,m_5,m_6\rangle=\sum_{\mathcal{R}_3,\beta_3}C^{\mathcal{R}_3,\beta_3*}_{m_1,m_2,m_3}C^{\mathcal{R}_3,\beta_3}_{m_4,m_5,m_6}\times V_3(\mathcal{R}_3,\beta_3). \tag{18}$$

For $V_3=W$ the above formula reduces to just one term. However, diagonalization of V_3 is still far more difficult than of a (also L-conserving) V_2 because of a larger number of nonzero CI matrix elements (by over 10 times in the systems discussed further in this section).

A. Moore-Read $\nu=\frac{1}{2}$ incompressible ground-state and excited magnetoroton bands

In Figs. 7(a) and 7(b) we present the results for $N=12$ and 14 and $2l=2N-3$. It is well known that for even values of N and for $2l=2N-3$ there is exactly one state in the spectrum with $E=0$, i.e., with no triplet amplitude at $\mathcal{R}_3=3$. In other words, the Hilbert subspace with $\mathcal{R}_3>3$ for all triplets contains exactly one state in this case. At $2l<2N-3$, all states have amplitudes at $\mathcal{R}_3=3$, and at $2l>2N-3$ there is more than one such state. For odd values of N, no such states occur for at $2l\leqslant 2N-3$, and at $2l>2N-3$ there are always more than one. This fact makes the Moore-Read state yet another beautiful extension of the Laughlin idea for the $\nu=\frac{1}{3}$ state at $2l=3N-3$ being the only state in its Hilbert space with no pair amplitude at $\mathcal{R}_2=1$. Just as the avoidance of more than one pair state generated the whole $\nu=\frac{1}{3},\frac{1}{5},\ldots$, sequence, the avoidance of not just pairs, but triplets (or K-body states) gives rise to incompressibility at new values of ν.

The analogy to the Laughlin $\nu=\frac{1}{3}$ state goes beyond the incompressible ground state. The low-energy excitations clearly form a band that resembles the magnetoroton curve.[31] In frame (c) we overlay data obtained for different $N=6$ to 14 and plotted as a function of wave vector k (the charge-neutral excitations carrying $L>0$ on a sphere move along great circles of radius R, but on a plane they would move along straight lines with $k=L/R$). The continuous character of this band and the minimum at $k \approx 1.5\,\lambda^{-1}$ (very close to $k \approx 1.4\,\lambda^{-1}$ of the Laughlin $\nu = \frac{1}{3}$ state) are clearly visible (similar curve has been shown previously;[31] here we only add a proper scaling with N).

B. Pairing and Laughlin pair-pair correlations

Before we move on to the spectra at $2l \neq 2N-3$ in search of the elementary charge excitations of the Moore-Read state, let us recall Halperin's[12] concept of Laughlin states of $\mathcal{R}_2 = 1$ pairs that we have also used earlier for the half filling of both LL_1 (Ref. 17) and CF-LL_1.[36,44] The increase of $\mathcal{G}_2(1)$ compared to a Laughlin-correlated state at the same ν visible in Fig. 3(a) can be thought of as pairing for both $\alpha \sim \frac{1}{2}$ and 1. However, whether the $\mathcal{R}_1$ pairs will keep far apart from one another by avoiding small values of their relative (pair-pair) angular momentum (what we would consider Laughlin correlations among the pairs) has not been established in either LL_1 or CF-LL_1. Actually, the fact that only for $N = 6, 10, 14, \ldots$ (and not for $N = 8$ or 12) do the $L = 0$ ground states occur in CF-LL_1 suggests that Halperin's idea could not be correct for the interacting QE's. However, for the half-filled LL_1, the occurrence of a large value of $\mathcal{G}_2(1)$ and, at the same time, the vanishing of $\mathcal{G}_3(3)$ finally offers support for this idea in the Moore-Read state. By effectively acting as a short-range three-body repulsion W, Coulomb repulsion in LL_1 allows grouping electrons into pairs (at ν as large as $\frac{1}{2}$), but it prevents the third electron from getting too close to a pair. As a result, the pairs exist but each pair attains a hard core that results in Laughlin correlation with all other pairs (or unpaired electrons), and that can be modeled by a fictitious flux attachment in a standard way.

Let us demonstrate how this picture works for the spectra in Fig. 7. As a result of the appropriate CF transformation,[17,36,44] N electrons at $2l = (2N+3) \pm \Delta$ are converted to $N_2 = \frac{1}{2}N$ CF's about exactly filling their effective CF-LL_0 shell with $2l_0^* = 2(2l-1) - 7(N_2-1) = (N_2-1) \pm 2\Delta$, i.e., with the effective degeneracy

$$g_0^* = N_2 \pm 2\Delta. \tag{19}$$

These CF's correspond to the $\mathcal{R}_2 = 1$ pairs of electrons, and their effective angular momentum l_0^* is obtained from $L_2 = 2l - 1$ by attachment of seven flux quanta to each pair (four to account for the pair-pair hard core due to Pauli exclusion principle, four to model pair-pair Laughlin correlations, and 1 in the opposite direction to convert the pairs to fermions). At exactly $2l = 2N-3$, the N-body (Moore-Read) ground state is equivalent to a full CF-LL_0 with $l_0^* = \frac{1}{2}(N_2-1)$, i.e., to a Laughlin state of N_2 pairs. The magnetoroton band describes QE-QH pair states, with one CF excited from the full CF-LL_0 to the empty CF-LL_1 with $l_1^* = l_0^* + 1 = \frac{1}{2}(N_2+1)$. This band extends up to $L = l_0^* + l_1^* = N_2$. Higher states above the magnetoroton band contain additional QE-QH pairs, and the characteristic steps are clearly visible in the energy spectra in Fig. 7 [e.g., at $L = (2l_0^*-1) + (2l_1^*-1) = N-2$ for two QE-QH pairs].

C. Quasiparticles

In Fig. 8 we present sample spectra obtained for even values of N and $2l = (2N-3) \pm 1$. As shown previously,[28,31] at $2l = (2N-3)+1$ there is always a band of $E = 0$ states at $L = N_2, N_2-2, \ldots$, corresponding to two QH's in CF-LL_0 of degeneracy $g_0^* = N_2 + 2$. This is shown in frame (a) for $N = 12$. Unlike for a Laughlin $\nu = \frac{1}{3}$ state of unpaired electrons, the increase of $2l$ by unity from the value corresponding to a full CF-LL_0 creates not one but two QH's, as predicted by Eq. (19) for the picture of Laughlin-correlated pairs. Note that the same is true for the finite-size Jain $\nu = \frac{2}{5}$ states with two CF LL filled; however, no combination $(N, 2l)$ corresponds to a single QH in a finite-size Moore-Read state (the condition $g_0^* = N_2 + 1$ leads to a half integral value of g), while for the Jain $\nu = 2/5$ state it occurs for even N, at $2l = \frac{1}{2}(5N-7)$. Similarly as in Fig. 7, the first excited band above the 2QH states contains an additional QE-QH pair, and it extends to $L = (3l_0^* - 3) + l_1^* = N$, exactly as marked in frame (a).

At $2l = (2N-3) - 1$ no states can have $E = 0$, but the lowest band is expected to contain two QE's in CF-LL_1 of degeneracy $g_1^* = N_2$. Indeed, in spectra (b) and (c) obtained for $N = 14$ and 16, the low-energy bands at $L = N_2 - 2, N_2 - 4, \ldots$, can be found as expected (although they are not as well resolved as the QH bands).

What is the electric charge $\mathcal{Q}$ of the QE's and QH's? Being proportional to the LL degeneracy, it can be obtained from the ratio of g^* and $g = N/\nu$ calculated in the $N \to \infty$ limit. For a Jain $\nu = n/(2pn+1)$ state of n completely filled CF-LL's, the degeneracy of each one is $g^* = N/n$, which leads to the well-known result $\mathcal{Q}/e = g^*/g = (2pn+1)^{-1}$. For the present case, $g = 2N$, $g^* = N_2 = \frac{1}{2}N$, and the result is precisely what should be expected for a $\nu = \frac{1}{8}$ state of $2e$-charged boson pairs[12]

$$\mathcal{Q} = e/4. \tag{20}$$

D. Spectra for odd particle numbers

If the Halperin[12] picture could be simply extended to finite $\nu = \frac{1}{2}$ systems with odd electron numbers N, they would contain $N_2 = \frac{1}{2}(N-1)$ pairs and $N_1 = 1$ unpaired electron, forming a two-component Laughlin-correlated fluid.[17] What actually happens is quite different,[27] as shown in our two sample energy spectra in Figs. 9(a) and 9(b).

At $2l = (2N-3)+1$ there is a band of $E = 0$ states that indeed correspond to a pair of QH's of the two-component fluid. In the CF picture, each QH has $l_0^* = \frac{1}{2}(N_2+1)$ which gives the total $L = N_2, N_2 - 2, \ldots$, exactly as obtained for $N = 11$ in frame (a).

At $2l = 2N-3$ no $E = 0$ states occur, and the numerical results for different N always show a band at $L = \frac{5}{2}, \frac{7}{2}, \ldots, \frac{1}{2}N$, that seems to describe dispersion of an excitonic state of a pair of QP's of opposite charge. This becomes more convincing in Fig. 9(c), where the data obtained for different N is (for the first time) plotted together as a function of wave vector k, and a clear magnetoroton-type minimum appears at $k \approx 1.0\lambda^{-1}$. The values $l = \frac{1}{4}(N \pm 5)$ of the QP angular mo-

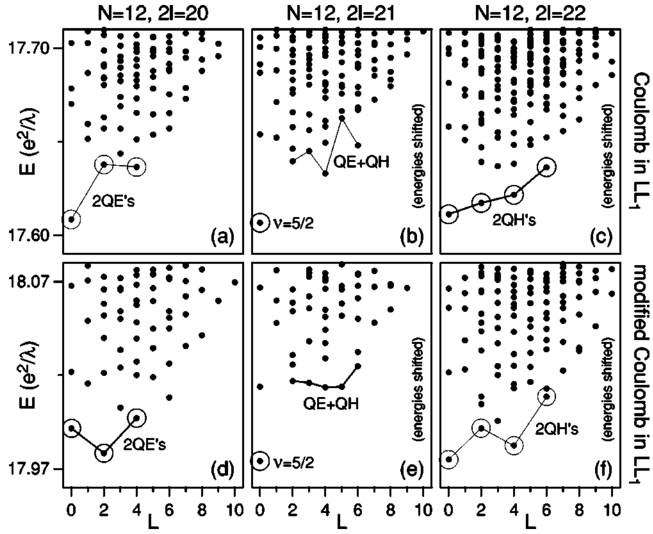

FIG. 10. (a)–(c) Energy spectra similar to those in Figs. 7–9, but obtained for the Coulomb pair pseudopotential of the first excited LL in Fig. 1(b). (d)–(f) Spectra of the same systems obtained for parameter $V_2(1)$ increased by 9%.

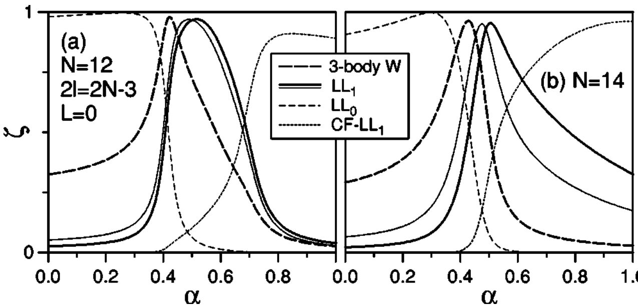

FIG. 11. Squared overlaps ζ of the lowest $L=0$ eigenstate of pair interaction U_α defined by Eq. (5) calculated on a sphere at $2l=2N-3$, with the corresponding eigenstates of three-body repulsion W (Moore-Read state), electron interaction in the lowest and excited LL (the narrow solid line is for layer width $w=3.5\lambda$), and QE interaction in the Laughlin $\nu=\frac{1}{3}$ state, plotted as a function of α. Frames (a) and (b) correspond to $N=12$ and 14 particles.

menta that would explain the observed range of L do not agree with the prediction of a Laughlin-correlated state with $N_2=\frac{1}{2}(N-1)$ and $N_1=1$. Nevertheless, knowing their angular momenta is enough to predict the charge $\pm e/4$ of these (unidentified) QP's.

The reason why this low-energy band cannot be described by a two-component CF model (actually, for any combination of N_1 and N_2) is that they are not pair-pair or pair-electron, but pair-breaking (PB) excitations (that by definition do not conserve N_1 or N_2) introduced by Greiter *et al.*[27] and later studied in detail by Bonesteel.[35] Such excitations generally occur in paired systems and they are expected to be charge-neutral (despite being fermions) which explains their continuous energy dispersion in a magnetic field. Still, it might be possible to decompose them into two more elementary, charged QP's.

VI. RELEVANCE TO THE $\nu=\frac{5}{2}$ FQH STATE

Earlier diagonalization studies[17,26,27,37] using Coulomb pseudopotential in LL_1 showed the $L=0$ ground states with a gap at $2l=2N-3$ but no clear indication of QP excitations identified[31] in the spectra of the model three-body repulsion W. As shown in the top frames of Fig. 10 obtained for $N=12$ electrons, the magnetoroton QE-QH band and of the two-QE bands can indeed hardly be found in these spectra due to mixing with higher states, and only the two-QH bands are well separated.

The problem with the identification of the Coulomb $\nu=\frac{1}{2}$ ground state in LL_1 with the Moore-Read (or any other) trial state is that the former is very sensitive to the relative values of the leading pseudopotential coefficients, while the exact form of $V_2(\mathcal{R}_2)$ depends (at least, in principle) on the layer width w in experiments, and on N in finite-size calculations. As to the width dependence, it turns out that increasing w from zero to realistic experimental values only weakly affects the nearly harmonic behavior of $V_2(\mathcal{R}_2)$ at short range that is essential for the avoidance of the $\mathcal{R}_3=3$ triplet state. As a result, the $\nu=\frac{5}{2}$ wave function in experimental systems depends much less on the width than, e.g., the excitation gap controlled by the magnitude of V_2.

On the other hand, the strong dependence of correlations on $\alpha\sim\frac{1}{2}$ in finite systems is clear in Figs. 3(a), 5(a), and 6, and it is in contrast with the behavior at $\alpha\sim 0$ or 1, corresponding to the much less sensitive finite-size FQH states in LL_0 and CF-LL_1. Remarkably, the gap above the incompressible ground state at $2l=2N-3$ persists[36] over a wide range of α despite even a large distortion of its wave function, while the QP excitations quickly mix with the continuum of higher states when V_2 becomes too subharmonic or superharmonic at short range.

A major problem with the calculations on a sphere is the size dependence (16) of the critical value of α at which the avoidance of $\mathcal{R}_3=3$ occurs. It is clearly visible in the plots of squared overlaps $\zeta_u(\alpha)=|\langle\phi_\alpha|\psi_u\rangle|^2$ with the eigenstates ϕ_α of U_α, calculated for the corresponding eigenstates ψ_u of various other interactions u: three-body repulsion W and electron and QE pair pseudopotentials V_2 in LL_0, LL_1, and CF-LL_1, respectively. For LL_1, the overlaps ζ_{LL1} have been calculated for both narrow ($w=0$) and wide ($w=3.5\lambda$; e.g., $w=20$ nm at $B=20$ T) layers. Note also that the eigenstate of W used in the calculation of overlaps is automatically properly symmetrized (in the original form[25] it is not[26]).

In Fig. 11 we plot the overlaps for the lowest $L=0$ states at $2l=2N-3$. Clearly, the exact Moore-Read eigenstate of W is an excellent ground state of U_α at $\alpha\approx 0.425$. So is the ground state of Coulomb pair interaction in LL_1, but at a different $\alpha\approx 0.5$. The disagreement between these two values of α does not disappear in wide samples, as inclusion of w even as large as 3.5λ does not noticeably change the Coulomb $\nu=\frac{5}{2}$ ground state. Specifically, the overlaps between the Moore-Read state and the Coulomb $\nu=\frac{5}{2}$ ground state calculated for $N=14$ are only $|\langle\psi_W|\psi_{C1}\rangle|^2=0.48$, 0.58, and 0.71 for $w/\lambda=0$, 1.75, and 3.5, respectively

The behavior of $\zeta_{QE}(\alpha)$ plotted with narrow dotted lines is also noteworthy. The QE-QE interaction at half filling can be

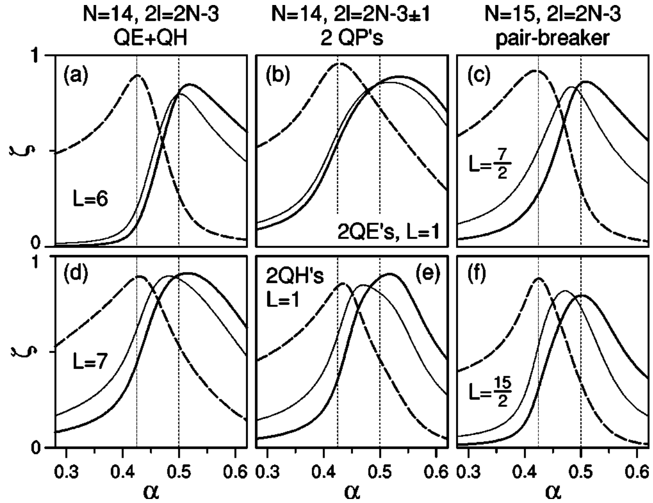

FIG. 12. Similar to Fig. 11 (solid and dashed lines mean the same) but for different low-energy states at $2l=(2N-3)$ and $(2N-3)\ \pm 1$, corresponding to pair QE and QH states and the pair-breaking excitation of the three-body interaction W.

described by U_1 quite well for $N=14$ (where the calculations indicate a finite-size $L=0$ ground state with a gap) and somewhat worse for $N=12$ (where the ground state is compressible). But even more interestingly, the Moore-Read state appears nearly orthogonal to the QE states (the exact value for $N=14$ is $|\langle\psi_W|\psi_{\rm QE}\rangle|^2=0.03$), which we interpret as yet another strong indication against the QE pairing at $\nu=\frac{3}{8}$.

In Fig. 12 we plot similar overlaps calculated for various excitations. Frames (a), (d) correspond to a QE-QH pair, (b) to two QE's, (e) to two QH's, and (c), (f) to the PB neutral-fermion excitation. We only show the curves for the QE-QH states at $L=6$ and 7 near the magnetoroton minimum, for two-QE and -QH states at small $L=1$ (corresponding to large QP-QP separation for which the curves are less dependent on QP-QP interaction effects), and for the PB at $L=\frac{7}{2}$ near the energy minimum and at a large $L=\frac{15}{2}$. All frames show similar behavior to Fig. 11, only the disagreement between the eigenstates of W and the Coulomb eigenstates is more pronounced. The QP excitations of the three-body repulsion W remarkably well describe actual excitations of a system with a two-body interaction U_α. However, not for the value of α corresponding to the Coulomb interaction in LL_1 (regardless of the layer width). The overlaps between eigenstates of W and the electron eigenstates in LL_1 are even lower than those for the Moore-Read state. The specific values for $N=14$ and $w=0$ (and for $w=3.5\lambda$ in parentheses) are $|\langle\psi_W|\psi_{C1}\rangle|^2=0.03$, 0.00, 0.27, 0.19, 0.12, 0.46 (0.03, 0.02, 0.39, 0.31, 0.20, 0.60) for the $L=2,3,\ldots,7$ states of the magnetoroton QE+QH band, 0.47, 0.16, 0.07 (0.52, 0.28, 0.14) for the $L=1$, 3, 5 states of two QE's, and 0.39, 0.12, 0.39, 0.27 (0.53, 0.17, 0.64, 0.32) for the $L=1$, 3, 5, 7 states of two QH's, respectively. The values for the PB band for $N=13$ are 0.45, 0.19, 0.41, 0.31, 0.34 (0.56, 0.34, 0.44, 0.46, 0.47) for $L=\frac{5}{2},\frac{7}{2}\ldots,\frac{13}{2}$, respectively. Such small overlaps preclude (indicated) interpretation of excited states in Fig. 10(a) and 10(b) as QE's of W .

This invokes the question raised in the introduction of whether the Moore-Read trial state and its QP excitations are only an elegant idea, not realized in known even-denominator FQH states (at $\nu=\frac{5}{2}$ or $\frac{3}{8}$). Fortunately, the disagreement appears to be largely artificial. The size dependence (16) of α_0 can be traced to the size dependence of the pair amplitudes $\mathcal{G}_2(\mathcal{R}_2)$ of the triplet eigenstates at $\mathcal{R}_3=3,5,6,\ldots$, directly caused by the surface curvature. This is obtained by combining the following observations: (i) the occurrence of "$\mathcal{R}_3>3$" three-body correlations defining the Moore-Read state depends not directly on the specific short-range behavior of pair pseudopotential V_2, but on the form of triplet pseudopotential V_3 at small $\mathcal{R}_3$; (ii) the relation between V_2 and V_3 (on a sphere) is more size sensitive than the short-range harmonic behavior of Coulomb interaction V_2. It is therefore only due to the surface curvature that (in finite systems on a sphere) $\alpha_0<\frac{1}{2}$ is different from the value $\alpha=\frac{1}{2}$ appropriate for the Coulomb pseudopotential in LL_1. This is consistent with larger overlaps calculated for the Moore-Read state on a torus.[26]

At $N\to\infty$, we expect that $\alpha_0\approx\frac{1}{2}$ in coincidence with the behavior of $V_2(\mathcal{R}_2)$ in the same limit, and that the energy spectra of Coulomb V_2 and model W interactions should become similar. To improve the agreement at $N\leqslant 14$, for which we were able to calculate the spectra, $V_2(1)$ must be slightly enhanced in accordance with Eq. (16). For example, for $N=12$ the near vanishing of $\mathcal{G}_3(3)$ at $2l=2N-3$ occurs when $V_2(1)$ is increased by 9% from its Coulomb value, in good agreement with the result of Morf.[37] The $N=12$ electron energy spectra calculated for this interaction with marked features associated with the QP's are shown in bottom frames of Fig. 10.

The above discussion yields the following statements. (i) Finite-size calculations on a sphere using Coulomb pair interaction do not correctly reproduce correlations of an infinite $\nu=\frac{5}{2}$ state. They use pseudopotentials corresponding to $\alpha\approx\frac{1}{2}$, different from $\alpha_0<\frac{1}{2}$ leading to the avoidance of $\mathcal{R}_3=3$. The $\alpha=\alpha_0=\frac{1}{2}$ coincidence is probably recovered for $N\to\infty$ which would mean that the real, infinite systems at $\nu=\frac{5}{2}$ do have the "$R_3>3$" correlations while the correlations in finite systems are different and size dependent. (ii) In finite systems, correct "$R_3>3$" correlations are recovered if the pair pseudopotential is appropriately enhanced at short range. (iii) Assuming that that the $\alpha=\alpha_0=\frac{1}{2}$ coincidence is restored in infinite systems (or in different, e.g., toroidal geometry), the equivalence of Coulomb and W interactions at half filling is not limited strictly to the Moore-Read ground state. The $(\pm e/4)$-charged QP's and the neutral-fermion PB identified in the spectra of W accurately describe the low-energy charge excitations in the real (Coulomb) $\nu=\frac{5}{2}$ systems. Although the effective interactions between QP's may lead to their binding or dressing (just as at $\nu=\frac{1}{3}$ QH's and "reversed-spin" QE's bind to form skyrmions), they are simple objects with an elegant interpretation in terms of Laughlin-like three-body correlations.

VII. CONCLUSION

We have studied two- and three-body correlations in partially filled degenerate shells for various interactions between the particles. Variation of the relative strength of two leading pair pseudopotential coefficients drives the correlations through three distinct regimes. The intermediate regime, corresponding to the nearly harmonic pseudopotential at short range, describes correlations among electrons in LL_1, particularly in the $\nu=\frac{5}{2}$ FQH state.

In contrast to the correlations between electrons in LL_0 or between Laughlin QE's in CF-LL_1 (whose pseudopotentials are strongly superharmonic and subharmonic at short range, respectively), the intermediate regime is not characterized by a simple avoidance of just one pair eigenstate corresponding to the strongest anharmonic repulsion. Instead, we have shown that near half filling the low-energy states for such interactions have simple three-body correlations. In resemblance of Laughlin pair correlations, they consist of the maximum avoidance of the triplet state with the smallest relative angular momentum $\mathcal{R}_3=3$, i.e., with the smallest area spanned by the three particles (in analogy to pair correlations, avoidance means here the minimization of a triplet amplitude).

In particular, at exactly half filling, this corresponds to the fact[27] that the Moore-Read ground state is the zero-energy eigenstate of a model short-range three-body repulsion W with the only pseudopotential parameter at $\mathcal{R}_3=3$. The Moore-Read ground state is a three-body analog of the Laughlin $\nu=\frac{1}{3}$ state with $\mathcal{R}_2>1$. It is separated by a finite excitation gap from a magnetoroton band with a minimum at $k\approx 1.5\lambda^{-1}$. Its elementary excitations are the $(\pm e/4)$-charged QP's (Refs. 27 and 31) (that naturally occur for Halperin[12] state) and the PB excitation.[27,35] The bands of few-QP states near half filling are well described by a CF picture appropriate for Laughlin pair-pair correlations.

Finally, the problem of numerical calculations on a sphere associated with the surface curvature is addressed. It is found that finite-size models using Coulomb interaction between electrons do not correctly reproduce correlations of the $\nu=\frac{5}{2}$ FQH state due to the distortion of triplet wave functions. Especially for the excitations of the $\nu=\frac{5}{2}$ ground state, the overlaps with the Moore-Read-like correlated states are rather small. However, it is argued that the $\nu=\frac{5}{2}$ FQH state observed experimentally in narrow systems is described much better by the Moore-Read trial state than could be inferred from the small-size calculations. Consequently, the origin of its incompressibility is precisely the avoidance of the $\mathcal{R}_3=3$ triplet state, and its elementary excitations are the $(\pm e/4)$-charged QP's and the neutral PB.

ACKNOWLEDGMENTS

This work was supported in part by Grant No. DE-FG 02-97ER45657 of the Materials Science Program - Basic Energy Sciences of the U.S. Department of Energy. A.W. acknowledges support by the Polish Ministry of Scientific Research and Information Technology under Grant No. 2P03B02424.

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