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The Fermion–Boson transformation in fractional quantum Hall systems

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Abstract

A Fermion-to-Boson transformation is accomplished by attaching to each Fermion a single flux quantum oriented opposite to the applied magnetic field. When the mean field approximation is made in the Haldane spherical geometry, the Fermion angular momentum l_{F} is replaced by $l_{\mathrm{B}} = l_{\mathrm{F}} - \frac{1}{2}(N-1)$. The set of allowed total angular momentum multiplets is identical in the two different pictures. The Fermion and Boson energy spectra in the presence of many-body interactions are identical if and only if the pseudopotential is "harmonic" in form. However, similar low-energy bands of states with Laughlin correlations occur in the two spectra if the interaction has short range. The transformation is used to clarify the relation between Boson and Fermion descriptions of the hierarchy of condensed fractional quantum Hall states.

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1. Introduction

The transformation of electrons into composite Fermions (CF) by attaching to each particle a flux tube carrying an even number of flux quanta has led to a simple intuitive picture [1] of the fractional quantum Hall effect (FQHE) [2]. Shortly after the introduction of the CF picture, Xie et al. [3] introduced a Fermion→Boson (F→B) mapping connecting a 2D Fermion system at filling ν_{F} with a 2D Boson system at filling ν_{B}, where $\nu_{\mathrm{F}}^{-1} = \nu_{\mathrm{B}}^{-1} + 1$. These authors stated that the sizes of the many-body Hilbert spaces for the Boson and Fermion systems were identical, and that their numerical calculations verified that the mapping accurately transformed the ground state of the Fermion system into the ground state of the

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Boson system if and only if these ground states were incompressible FQH states. In this paper we show that the F→B transformation leads to identical energy spectra if and only if the pseudopotential $V(L_{12})$ describing the interactions among the particles is of the "harmonic" form $V_H(L_{12}) = A + BL_{12}(L_{12}+1)$, where A and B are constants, and L_{12} is the total angular momentum of the interacting pair [4]. Laughlin correlations [5] occur when the actual pseudopotential $V(L_{12})$ rises more quickly with increasing L_{12} than $V_H(L_{12})$. Anharmonic effects (due to $\Delta V(L_{12}) = V(L_{12}) - V_H(L_{12})$) cause the interacting Fermion and interacting Boson spectra to differ for every value of the filling factor $\nu_F = \nu_B(1+\nu_B)^{-1}$. However, for appropriately chosen (short-range) model pseudopotentials, the F→B mapping accurately transforms the ground state of the Fermion system to that of the Boson system both for incompressible FQH states and for other low-lying states. The F→B mapping is also very useful in understanding the relation between the Haldane [6] hierarchy of Boson quasiparticle (QP) condensates and the CF hierarchy [7,8] of Fermion QP condensed states.

2. Gauge transformations in two-dimensional systems

By attaching to each Fermion or Boson of charge $-e$, a fictitious flux tube carrying an even number $2p$ of flux quanta oriented opposite to the applied field, the eigenstates and particle statistics are unchanged. The "gauge field" interactions between the charge on one particle and the vector potential due to the flux quanta on every other particle make the Hamiltonian more complicated. Only when the mean field approximation is made does the problem simplify. In addition to these (CF and CB) transformations, a F→B transformation can be made by attaching to each Fermion an odd number $2p+1$ of flux quanta (one flux quantum changes the statistics; other $2p$ flux quanta describe an additional CF or CB transformation). If the particles are confined to the surface of a sphere containing at its center a magnetic monopole of strength $2S_F$ (for Fermions) or $2S_B$ (for Bosons) flux quanta, then the lowest shell of mean field composite particles has angular momentum $l_F^* = |l_F - p(N-1)|$ where $l_F = S_F$ or $l_B^* = |l_B - p(N-1)|$ where $l_B = S_B$. In the F→B transformation (with $p = 0$), l_F is replaced by $l_B = |l_F - \frac{1}{2}(N-1)|$.

3. Some useful theorems

When a shell of angular momentum l contains N identical particles (Fermions or Bosons), the resulting N particle states can be classified by eigenvectors $|L, M, \alpha\rangle$, where L is the total angular momentum, M its z-component, and α a label which distinguishes independent multiplets with the same total angular momentum L. In the mean field CF (CB) transformation l_F (l_B) is transformed to l_F^* (l_B^*). In trying to understand why the mean field CF picture correctly predicted the low-lying band of states in the interacting electron spectrum, the following theorem was important [9,10].

Theorem 1. *The set of allowed total angular momentum multiplets of N Fermions each with angular momentum* l_F^* *is a subset of the set of allowed multiplets of N Fermions each with angular momentum* $l_F = l_F^* + (N-1)$.

Thus, if we define $g_{Nl}(L)$ as the number of independent multiplets of total angular momentum L formed by addition of the angular momenta of N Fermions, each with angular momentum l, then $g_{Nl^*}(L) \leqslant g_{Nl}(L)$ for every value of L. A few examples for small systems suggest that the theorem is correct, but a general mathematical proof is non-trivial. A proof using the methods of combinatorics and the KOH [11,12] theorem has been given recently [13].[1] The same method allows the proof of a second theorem.

Theorem 2. *The set of allowed total angular momentum multiplets of N Bosons each with angular momentum* l_B *is identical to the set of multiplets for N Fermions each with angular momentum* $l_F = l_B + \frac{1}{2}(N-1)$ *[13] (see Footnote 1).*

From Theorem 2, it follows immediately that Theorem 1 also applies to Bosons. Theorem 2 is a stronger

[1] The second theorem has been arrived at independently by B. Wybourne, private communication.

statement than a simple equality of the sizes of the many body Hilbert spaces [3].

4. Interaction effects

In studying why the mean field CF picture correctly predicts the low-lying states of a 2D electron system in a magnetic field [4,9], the "harmonic pseudopotential"

$$V_{\mathrm{H}}(L_{12}) = A + B\hat{L}_{12}^2 \tag{1}$$

was introduced. Here A and B are constants and $\hat{L}_{12}$ is the total angular momentum operator of the pair of particles. It was shown that for the harmonic pseudopotential the energy of any multiplet of angular momentum L was given by

$$E_{L\alpha} = A \cdot \tfrac{1}{2} N(N-1) + B \cdot N(N-2)\, l(l+1) + B \cdot L(L+1). \tag{2}$$

The energy is independent of α, so that every multiplet with the same value of L has the same energy. Eq. (2) holds both for Fermions and for Bosons. If $B_{\mathrm{F}} = B_{\mathrm{B}} = B$, then the spectrum of N Bosons each with angular momentum l_{B} is identical (up to a constant) to that of N Fermions each with angular momentum $l_{\mathrm{F}} = l_{\mathrm{B}} + \frac{1}{2}(N-1)$. This is a rather surprising result because Fermions and Bosons sample different sets of values of the pair angular momentum. For example, for $N = 9$ and $l_{\mathrm{F}} = 12$ (corresponding to $\nu_{\mathrm{F}} = \frac{1}{3}$) the allowed values of the Fermion pair angular momentum consist of all odd integers between 1 and 23; for the corresponding Boson system with $l_{\mathrm{B}} = 8$ ($\nu_{\mathrm{B}} = \frac{1}{2}$), the allowed values of L_{12} are all even integers between 0 and 16. Despite the totally different set of pseudopotential coefficients sampled, up to a constant, the spectra of the Boson and Fermion systems interacting through a harmonic pseudopotential are the same.

In earlier work [9], it was emphasized that the harmonic pseudopotential led to an "anti-Hund's rule" with the lowest energy state having the lowest allowed value of L. It is the positive anharmonicity $\Delta V(L_{12}) > 0$ that causes Laughlin correlations. It is useful to introduce the "relative" angular momentum $\mathscr{R} = 2l - L_{12}$. For Bosons $\mathscr{R}_{\mathrm{B}} = 0, 2, 4, \ldots$ while for Fermions $\mathscr{R}_{\mathrm{F}} = 1, 3, 5, \ldots$; in both cases, $\mathscr{R} \leqslant 2l$. We can write the pseudopotential and its harmonic and anharmonic parts in terms of $\mathscr{R}$, and call them $V(\mathscr{R})$, $V_{\mathrm{H}}(\mathscr{R})$, and $\Delta V(\mathscr{R})$, respectively. It is more reasonable to make simple models for $\Delta V(\mathscr{R})$ (e.g. assume that it vanishes for all $\mathscr{R}$ greater than some value) than for $V(\mathscr{R})$ itself. From Eq. (2) and the equation for the total energy,

$$E_{L\alpha} = \frac{1}{2} N(N-1) \sum_{\mathscr{R}} \mathscr{G}_{L\alpha}(\mathscr{R})\, V(\mathscr{R}), \tag{3}$$

where $\mathscr{G}_{L\alpha}(\mathscr{R})$ is the coefficient of fractional grandparentage (CFGP) [4,9,14], it is readily ascertained that the interacting Boson and interacting Fermion systems cannot have identical spectra when $\Delta V(\mathscr{R})$ is non-zero.

Xie et al. [3] determined the Boson and Fermion eigenfunctions by exact numerical diagonalization for six particle systems connected through the F→B transformation. They then transformed the Boson eigenfunctions into Fermion eigenfunctions by multiplying them by $\prod_{i<j}(z_i - z_j)$, as required by the B→F transformation. The overlap of these transformed Boson eigenfunctions with the exact Fermion eigenfunctions was then evaluated. The overlap was quite close to unity for incompressible quantum fluid states when the full Coulomb interaction was used. A similar result was obtained for a model short-range interaction H_1 for which $V(\mathscr{R})$ vanished for $\mathscr{R} > 1$ and was equal to the Coulomb values at $\mathscr{R} = 0$ (for Bosons) or at $\mathscr{R} = 1$ (for Fermions). However, when the interaction was approximated by H_3 for which $V(\mathscr{R})$ vanished for $\mathscr{R} > 3$ and was equal to the Coulomb values at $\mathscr{R} = 0$ and 2 (for Bosons) or at $\mathscr{R} = 1$ and 3 (for Fermions), the overlap was considerably smaller. The reason appears to be that for Fermions H_3 is subharmonic at $\mathscr{R} = 3$, while for Bosons it is (marginally) superharmonic in the entire range of $\mathscr{R}$, and that for a subharmonic pseudopotential Laughlin correlations are not expected to occur [14]. By a subharmonic (superharmonic) behavior of $V(\mathscr{R})$ at a certain value $\mathscr{R}_0$ we mean that $V(\mathscr{R}_0)$ is larger (smaller) than a value $V_{\mathrm{H}}(\mathscr{R}_0)$ for which $V(\mathscr{R})$ would be harmonic (i.e., linear in $L_{12}(L_{12}+1)$) in the range $\mathscr{R}_0 - 2 \leqslant \mathscr{R} \leqslant \mathscr{R}_0 + 2$. We will later use an anharmonicity parameter x defined as $x(\mathscr{R}_0) = V(\mathscr{R}_0)/V_{\mathrm{H}}(\mathscr{R}_0)$; for the H_3 interaction, $x(3) = 1.3$ (for Fermions) and $x(2) = 0.8$ (for Bosons).

We have evaluated numerically the eigenstates of an eight electron system at $2S_{\mathrm{F}} = 19$–23 (these states correspond to Laughlin $\nu_{\mathrm{F}} = \frac{1}{3}$ states with zero, one, or two QPs) for a number of different pseudopoten-

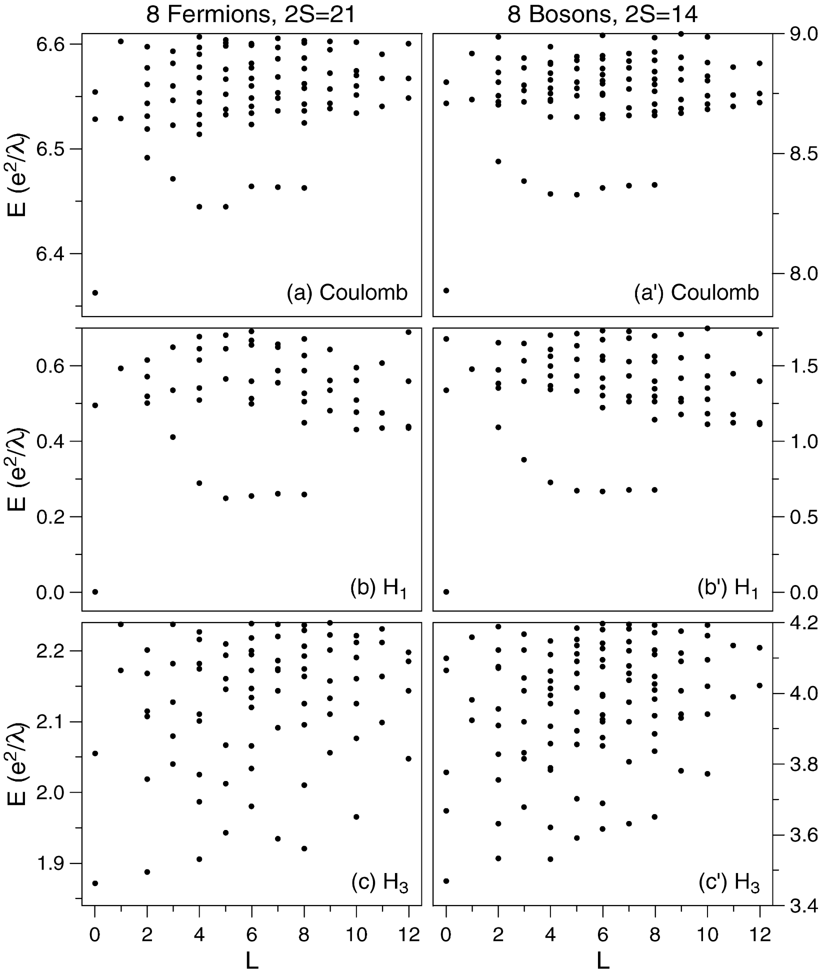

Fig. 1. The energy spectra (energy E versus angular momentum L) of the corresponding eight Fermion (left) and eight Boson (right) systems at the monopole strengths $2S_F = 21$ and $2S_B = 14$ (filling factors $\nu_F = \frac{1}{3}$ and $\nu_B = \frac{1}{2}$) for the Coulomb pseudopotential in the lowest Landau level $(a - a')$, and for the model pseudopotentials H_1 $(b - b')$, and H_3 $(c - c')$. λ is the magnetic length.

tials. We have used the full Coulomb pseudopotential, H_1, H_3, H_5, and a model pseudopotential V_x in which $V_x(1) = 1$, $V_x(\mathscr{R} \geqslant 5) = 0$, and $V_x(3) = x \cdot V_H(3)$ is an arbitrary fraction x of the "harmonic" value. We perform the same calculations for eight Boson systems at $2S_B = 12$–16 (here, $V_x(0) = 1$, $V_x(\mathscr{R} \geqslant 4) = 0$, and $V_x(2) = x \cdot V_H(2)$).

In Fig. 1, we contrast the energy spectra for the Fermion and Boson systems at $\nu_F = \frac{1}{3}$ ($\nu_B = \frac{1}{2}$) for the Coulomb pseudopotential appropriate for the lowest Landau level (a–a'), and for the model pseudopotentials H_1 $(b - b')$ and H_3 $(c - c')$. In Fig. 2, we do the same for the state containing two Laughlin quasielectrons (QE). The lowest states in $(a - a')$ and $(b - b')$ are quite similar consisting of a Laughlin $L = 0$ ground state in Fig. 1 and two-QE states with $l_{QE} = \frac{1}{2}(N - 1) = \frac{7}{2}$ giving $L = N - 2, N - 4, \ldots = 0, 2, 4,$ and 6 in Fig. 2. The magnetoroton band (at $2 \leqslant L \leqslant 8$) is apparent in Fig. 1 although the gaps and band widths are different for different pseudopotentials. The pseudopotential used in $(c - c')$ gives very different results both in Figs. 1 and 2. As mentioned before, this results because $V(3)$ used in Fermion pseudopotential H_3 is too large to lead to Laughlin correlations.

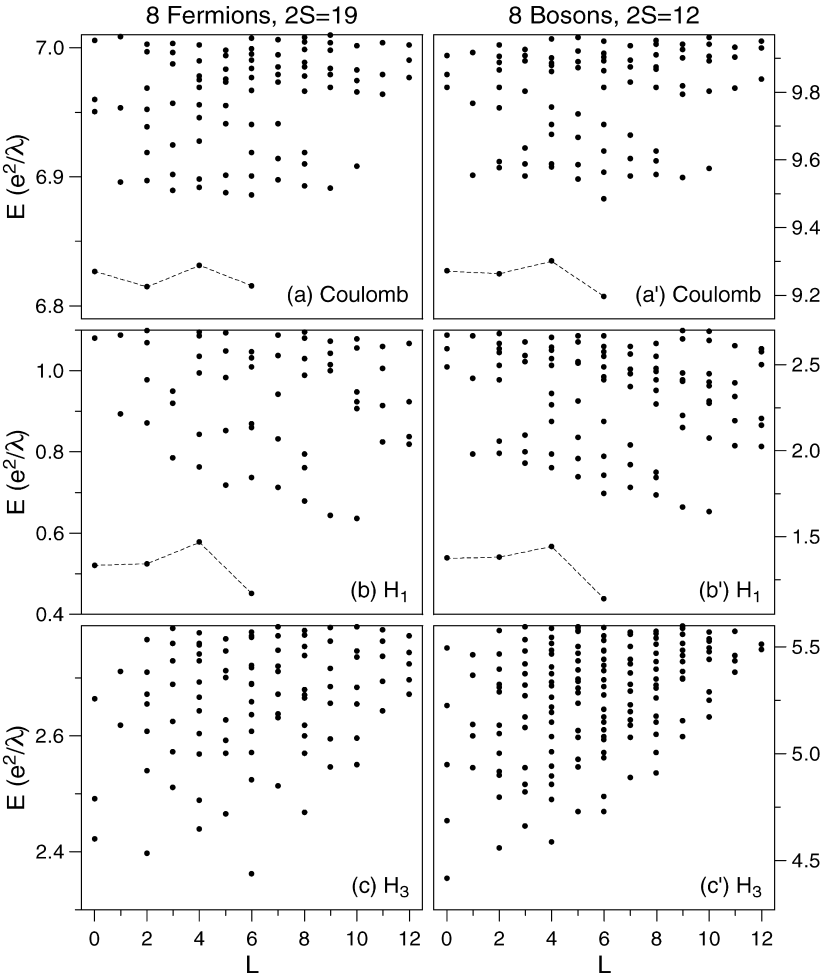

Fig. 2. The energy spectra (energy E versus angular momentum L) of the corresponding eight Fermion (left) and eight Boson (right) systems at the monopole strengths $2S_{\mathrm{F}} = 19$ and $2S_{\mathrm{B}} = 12$ (two Laughlin quasielectrons in the $\nu_{\mathrm{F}} = \frac{1}{3}$ and $\nu_{\mathrm{B}} = \frac{1}{2}$ state) for the Coulomb pseudopotential in the lowest Landau level ($a - a'$), and the model pseudopotentials H_1 ($b - b'$), and H_3 ($c - c'$). λ is the magnetic length.

To illustrate this point, we have calculated the energy spectra using pseudopotential V_x with different values of x. In Fig. 3, we show the spectra at $\nu_{\mathrm{F}} = \frac{1}{3}$ ($\nu_{\mathrm{B}} = \frac{1}{2}$) for $x = \frac{1}{2}, 1$, and $\frac{3}{2}$. For $x < 1$, $V_x(\mathscr{R})$ is superharmonic at $\mathscr{R} = 3$ (for Fermions; $x \equiv x(3)$) or at $\mathscr{R} = 2$ (for Bosons; $x \equiv x(2)$), and Laughlin correlations with an $L = 0$ ground state occur. For $x \geqslant 1$ there is little resemblance between the numerical spectra and that associated with the full Coulomb interaction. Furthermore, the Fermion and Boson spectra are quite different from one another.

From the eigenfunctions, we can determine CFGP's $\mathscr{G}_{L\alpha}(\mathscr{R})$ for each state $|L, \alpha\rangle$. In Fig. 4, we plot the x-dependence of the CFGP's $\mathscr{G}_{L\alpha}(\mathscr{R})$ from pair states at three smallest values of $\mathscr{R}$ calculated for the lowest energy $L = 0$ state of eight Fermions at $2S_{\mathrm{F}} = 21$ ($\nu_{\mathrm{F}} = \frac{1}{3}$) and eight Bosons at $2S_{\mathrm{B}} = 14$ ($\nu_{\mathrm{B}} = \frac{1}{2}$). In both systems, a Laughlin incompressible state with vanishing $\mathscr{G}(1)$ (for Fermions) or $\mathscr{G}(0)$ (for Bosons) occurs at small x, and a rather abrupt transition occurs at $x \approx 1$, implying a change of the nature of the correlations when the pseudopotential $V_x(\mathscr{R})$ changes from

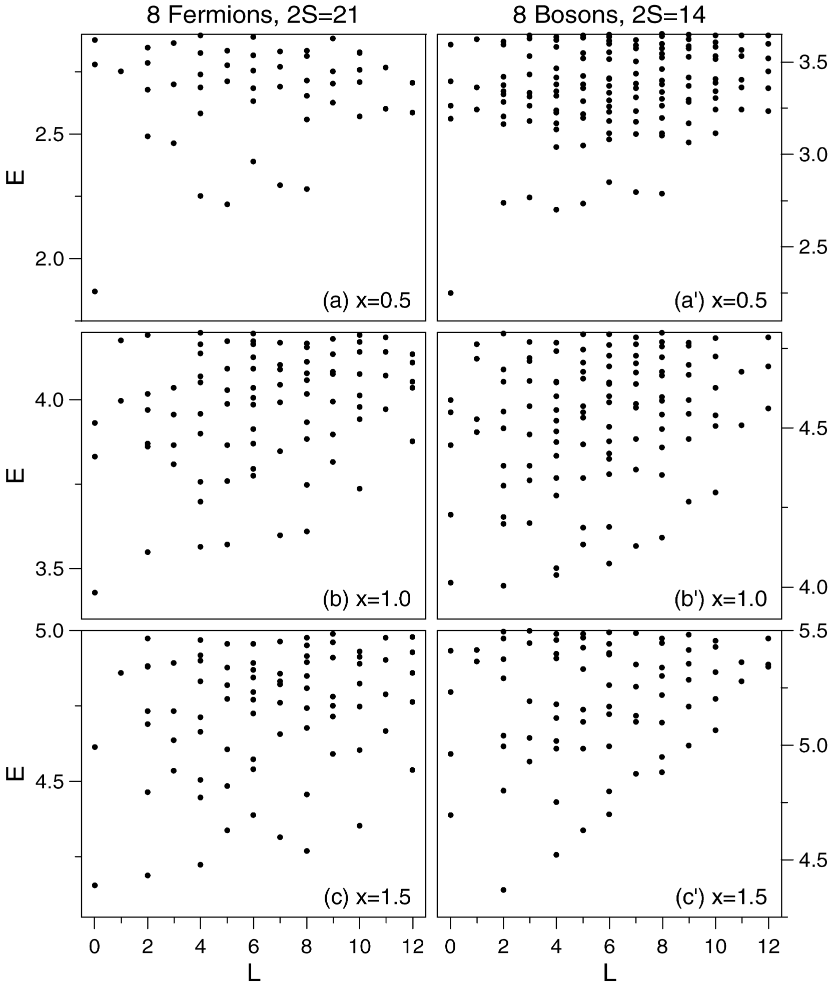

Fig. 3. The energy spectra (energy E versus angular momentum L) of the corresponding eight Fermion (left) and eight Boson (right) systems at the monopole strengths $2S_{\rm F}=21$ and $2S_{\rm B}=14$ (filling factors $\nu_{\rm F}=\frac{1}{3}$ and $\nu_{\rm B}=\frac{1}{2}$) for model interaction pseudopotentials $V_x(\mathscr{R})$ with $x=\frac{1}{2}$ $(a-a')$, $x=1$ $(b-b')$, and $x=\frac{3}{2}$ $(c-c')$.

super- to subharmonic. At $x>1$, the correlations in the two systems are quite different and, for example, another abrupt transition occurs in the Boson system at $x\approx 4$ (not shown in the figure), which is absent in the Fermion system.

5. Quasiparticles

The $\mathrm{F}\to\mathrm{B}$ transformation allows us to better understand the Boson [6] versus Fermion [7,8] description of QPs in incompressible FQH states. Laughlin condensed states having $\nu_{\rm F}=(2p+1)^{-1}$ (where p is a positive integer) occur at $2S_{\rm F}=(2p+1)(N-1)$ in the Haldane spherical geometry. The CF transformation [1] gives an effective angular momentum $l^*_{\rm F}=S^*_{\rm F}=S-p(N-1)=\frac{1}{2}(N-1)$ when $2p$ flux quanta are attached to each electron and oriented opposite to the applied magnetic field. Thus the N CFs fill the $2l^*+1$ states of the lowest CF shell giving an $L=0$ incompressible ground state.

The $\mathrm{F}\to\mathrm{B}$ transformation gives $2S_{\rm B}=2S_{\rm F}-(N-1)=2p(N-1)$ and a Boson filling factor of $\nu_{\rm B}=(2p)^{-1}$. Making a CB transformation gives

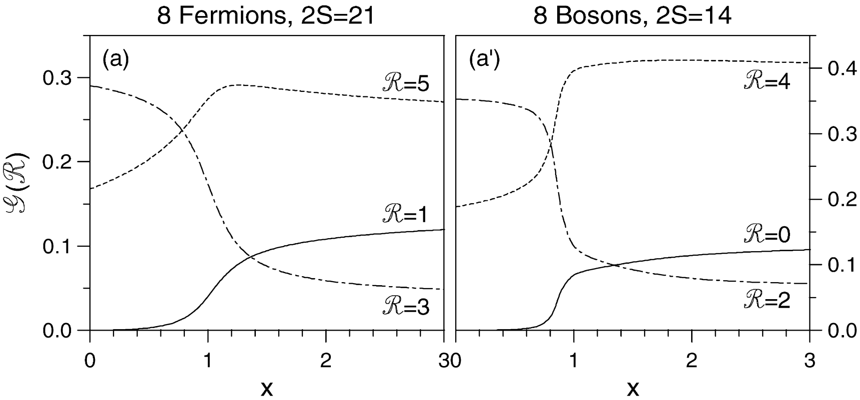

Fig. 4. The coefficients of fractional grandparentage $\mathcal{G}(\mathcal{R})$ from the pair states at three smallest values of $\mathcal{R}$ calculated for the lowest energy $L=0$ state of the corresponding eight Fermion (a) and eight Boson (a') systems at the monopole strengths $2S_{\mathrm{F}}=21$ ($\nu_{\mathrm{F}}=\frac{1}{3}$) and $2S_{\mathrm{B}}=14$ ($\nu_{\mathrm{B}}=\frac{1}{2}$) for the model interaction V_x, as a function of x.

$l_{\mathrm{B}}^*=S_{\mathrm{B}}^*=S_{\mathrm{B}}-p(N-1)=0$. This also gives an $L=0$ incompressible ground state because each CB has $l_{\mathrm{B}}^*=0$. Thus the CF description of a Laughlin state has one filled CF shell of angular momentum $l_{\mathrm{F}}^*=\frac{1}{2}(N-1)$, while the CB description has N CBs each with angular momentum $l_{\mathrm{B}}^*=0$.

For $2S_{\mathrm{B}}=2n(N-1)\pm n_{\mathrm{QP}}$, where the $+$ and $-$ occur for quasiholes (QH) and QE, respectively, we define $2l_{\mathrm{B}}^*=|2S_{\mathrm{B}}^*|=n_{\mathrm{QP}}$. This gives exactly the same set of angular momentum multiplets as obtained in the CF picture with $2S_{\mathrm{F}}=(2n+1)(N-1)+n_{\mathrm{QH}}$. However, it gives a larger set of multiplets than are allowed by $2S_{\mathrm{F}}=(2n+1)(N-1)-n_{\mathrm{QE}}$. For example, for $n_{\mathrm{QE}}=2$, $l_{\mathrm{B}}^*=1$ and the allowed values of the pair angular momentum of the two QPs are $N, N-2, N-4, \ldots$. For a Fermion system with $l_{\mathrm{F}}^*=\frac{1}{2}(N-1)-2$, the allowed values of the QP pair angular momentum are $N-2, N-4, \ldots$. The two sets can be made identical only if a hard core repulsion forbids the Boson QP pair from having the largest allowed pair angular momentum $L_{12}^{\mathrm{MAX}}=N$ [15]. This behavior is observed in Fig. 2, where the Boson treatment of two QEs (i.e., the CB transformation) would predict states at $L=0,2,4,6$, and 8, but the $L=8$ state does not occur in the low-energy band.

Since the description of CBs (with hard core QE interaction) and CFs give identical sets of QP states, filled QP levels (implying daughter states) occur at identical values of the applied magnetic field. In earlier work [7,8], we have emphasized that both the Haldane hierarchy and CF hierarchy schemes assume the validity of the mean field approximation, and we have shown that this approximation is expected to fail when the QP–QP interaction is subharmonic. Numerical results show when the mean field approximation is valid and when it fails.

6. Summary

We have shown that the $\mathrm{F} \to \mathrm{B}$ transformation replaces the single Fermion angular momentum l_{F} by $l_{\mathrm{B}}=l_{\mathrm{F}}-\frac{1}{2}(N-1)$, and that this transformation leads to an identical set of total angular momentum multiplets. The Fermion and Boson systems have identical spectra in the presence of many-body interactions only when the pseudopotential is harmonic, i.e. linear in squared pair angular momentum, $L_{12}(L_{12}+1)$. However, similar low-energy bands of states with Laughlin correlations occur in the two spectra if the interaction pseudopotential is superharmonic, i.e. has short range. We have studied numerically eight particle systems for different model interactions and shown the relation between the spectra and coefficients of fractional grandparentage for the Fermion and Boson systems. Finally, we have used the $\mathrm{F} \to \mathrm{B}$ transformation to clarify the relation between the Haldane Boson picture and the CF picture of the hierarchy of condensed states.

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