

Spin excitation spectra of integral and fractional quantum Hall systems

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Results are presented of detailed numerical calculations for the spin excitations of a two-dimensional electron gas confined to a quantum well (QW) of finite width w , at magnetic fields corresponding to both fractional and integral Landau level (LL) fillings $\nu \approx \frac{1}{3}, 1, 3$, and 5. The Hamiltonian matrix of up to $N=14$ interacting electrons is diagonalized exactly in Haldane spherical geometry, neglecting excitations to higher orbital LL's or to higher QW subbands. At sufficiently low Zeeman energy E_Z , spin waves (SW's) and skyrmions are found at $\nu = \frac{1}{3}$ that are composite fermion analogues of the reversed-spin-electron-hole excitations known to occur at $\nu = 1$. Only for sufficiently large w are stable skyrmions found for $\nu = 3$ and 5. Their stability depends upon the interaction pseudopotentials of their constituents. We propose a criterion on the pseudopotential needed for skyrmion stability, and construct phase diagrams (in the w - E_Z plane) for skyrmions of different size. The SW-SW and skyrmion-skyrmion interactions are also discussed, and the noninteracting SW condensates as well as the skyrmion fluids with Laughlin correlations (at intermediate skyrmion densities) are proposed.

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I. INTRODUCTION

The low-energy dynamics of a two-dimensional electron gas (2DEG) in a strong magnetic field B is determined by the particular form of the electron-electron (e - e) interaction in a macroscopically degenerate, partially filled Landau level (LL).¹⁻³ In the lowest LL, the short range of this (repulsive) interaction results in a particular form of (Laughlin) correlations.⁴ Namely, at each given density ρ , yielding a fractional LL filling $\nu = \rho \phi_0 / B = 2\pi \rho \lambda^2$ ($\phi_0 = hc/e$ is a magnetic flux quantum and $\lambda = \sqrt{\hbar c / eB}$ is the magnetic length), the electrons tend to avoid^{5,6} as much as possible (within their Hilbert space severely limited by the LL quantization) the pair eigenstates with the highest repulsion, that is with the smallest relative pair angular momentum $\mathcal{R} = 0, 1, 2, \dots$. This microscopic property is responsible for a variety of macroscopic, experimentally observable effects. A well-known example is the fractional quantum Hall effect^{4,7} in which a finite gap for charge excitations at $\nu = \frac{1}{3}, \frac{2}{5}$, etc., causes quantization of the Hall conductance at the universal values $\sigma_{xy} = \nu e^2 / h$. Another example is a non-linear dependence of the spin polarization ζ on the magnetic field B (or density) near $\nu = 1$ (in the systems in which the Zeeman energy E_Z can be made sufficiently small to allow spin excitations),⁸⁻¹⁴ related with the existence of particle-like excitations carrying massive spin—generally called skyrmions¹⁵⁻¹⁸ and equivalent to the topological solitons of the $O(3)$ nonlinear sigma (NL σ) model in two dimensions.¹⁹⁻²¹

Two main features determine the properties of the 2DEG in the quantum Hall regime: (i) the degenerate LL structure of the single-particle spectrum and (ii) the characteristic short-range repulsive interaction between the particles (electrons e or holes h) in a partially filled LL.¹ By the short

range e - e (or h - h) repulsion we mean here such that causes the tendency for the electrons (or holes) in the lowest-energy many-body states to avoid one or more pair eigenstates with the smallest average pair separation $\sqrt{\langle r^2 \rangle}$. This tendency can be considered a more general definition of Laughlin correlations than the occurrence of the Jastrow prefactor $\prod_{i < j} (z_i - z_j)^{2p}$ in the many body wave function, the latter being characteristic of the Laughlin $\nu = (2p + 1)^{-1}$ state only in the lowest LL.²⁸⁻³⁰

It is quite remarkable that both features (i) and (ii) repeat at different (integral or fractional) filling factors ν . The Laughlin correlations in a partially filled lowest LL allow (at low energy) the mapping of the original electron system near $\nu = (2p + 1)^{-1}$ onto the system of weakly interacting quasi-particles (QP's) with $(2p + 1)$ times smaller degeneracy g^* of their quasi-LL's.²² It is a matter of preference whether the reduced quasi-LL degeneracy $g^* = g / (2p + 1)$ is attributed to the fractional charge $\pm e / (2p + 1)$ of the QP's (Refs. 4 and 22) or, as in the composite fermion (CF) picture,²³⁻²⁶ to the partial cancellation of the magnetic field B by the Chern-Simons gauge field. In any case, the effective QP filling factor is $\nu^* \approx 1$ (which allows interpretation of the fractional quantum Hall effect of electrons as an integral quantum Hall effect²⁷ of QP's), with small residual interaction between the (charged) QP's.

The answer to the question of whether the low-energy spin excitations of all these similar systems are indeed similar (and, e.g., include stable skyrmions) lies in the details of this residual interaction within the relevant electron or CF LL. This question is not at all trivial. It is known, for example, that the electron correlations in partially filled higher LL's are qualitatively different from those in the lowest LL.²⁸⁻³⁰ Similarly, the short-range behavior of the interaction between QP's of the Laughlin $\nu = \frac{1}{3}$ state allows both quasiholes³¹ (QH's) and minority-spin quasielectrons³² (QE_R's) to form Laughlin $\nu_{\text{QP}} = \frac{1}{3}$ hierarchy states (corre-

sponding to $\nu = \frac{2}{7}$ and $\frac{4}{11}$, respectively), but forbids the formation of this state for majority-spin quasielectrons³¹ (QE's). To make things more complicated, the effective 2D interaction in realistic systems depends also on the details of the confinement (in the z direction, perpendicular to the 2D layer).³³

The validity of exact numerical studies of small systems in describing experiments is based on two observations. First, for problems in which short-range correlations play dominant role, the system size must only exceed the correlation length. Second, the qualitative behavior of the relevant QP-QP interactions as a function of \mathcal{R} is independent of N , the number of electrons, and the pseudopotential⁵ $V_{\text{QP-QP}}(\mathcal{R})$ converges rather quickly as N^{-1} is decreased. Moreover, for the physical situations in which neither screening nor disorder play an important role, excitations to higher LL's can be neglected. Another common approximation used in this paper is the confinement of the electrons and holes to their lowest QW subbands, in narrow QW's justified by the large excitation energy in the z direction $\hbar^2/2mw^2$.

The spin excitations at $\nu = 1$ have been investigated both experimentally⁸⁻¹⁴ and theoretically^{19-21,34-49} and not only in an extended 2DEG but also in finite-size quantum Hall droplets.⁵⁰⁻⁵² It is noteworthy that an exact mapping between the unpolarized electron ($\uparrow\downarrow$) and polarized electron-valence-hole (e - h) systems in the lowest LL (Ref. 53) makes a skyrmion S_K at $\nu = 1$, consisting of K spin waves bound to a spin- \uparrow electron or to a spin- \downarrow hole, equivalent to a charged multiexciton^{54,55} (X_K^\pm), consisting of K neutral excitons bound to a conduction electron or to a valence hole. This mapping relies on the fact that the electron (\uparrow and \downarrow) and hole LL bands are both dispersionless and thus equivalent despite different electron and hole effective masses. The mapping allows cross interpretation of the results of the (experimental and theoretical) studies of spin and optical excitations.

Unlike in the integral quantum Hall regime, skyrmions at $\nu = \frac{1}{3}$ (proposed by Kamilla *et al.*⁵⁶) were only recently detected in a transport experiment,^{57,58} thanks to a sufficient reduction of the Zeeman gap E_Z by means of hydrostatic pressure. Subsequent numerical calculation⁵⁹ also indicated the formation of small skyrmions and antiskyrmions in finite-size fractional quantum Hall systems. However, the question of why similar spin excitations occur in the $\nu = 1$ electron system and the $\nu^* = 1$ CF system despite different e - e and CF-CF interactions³¹ has not yet been answered.

The situation in higher LL's is not yet completely understood, either. First, it was predicted^{60,61} that skyrmions do not occur at $\nu = 3, 5$, etc. This prediction was soon confirmed experimentally.⁶² Then it was found⁶³ that a finite width of a quasi-2DEG stabilizes skyrmions in higher LL's. Indeed, rapid spin depolarization around $\nu = 3$ was recently observed⁶⁴ in a rather wide, 30 nm quantum well. While it is clear that the finite width enters the problem of an isolated, higher LL only through the weakening of the e - e repulsion at short range, the class of interactions for which skyrmions are stable has not yet been generally defined.

This article is an extended version of our short note.⁶⁵ We compare the results of detailed numerical calculations of the

spin excitation spectra at $\nu = \frac{1}{3}, 1, 3$, and 5. We identify spin waves and skyrmions in each system, and analyze their single-particle properties as well as their mutual interactions. Very similar spectra are obtained at $\nu = \frac{1}{3}$ and 1, and, if the layer width w is sufficiently large, also at $\nu = 3$ and 5. We confirm that skyrmions are the lowest-energy charged excitations in all these systems if the Zeeman energy E_Z is sufficiently small.

We show that the formation of particlelike skyrmions in some systems but not in others (depending on the filling factor, well width, and interactions) coincides with the occurrence of the condensed states of a macroscopic number of nearly noninteracting charge-neutral spin waves each with angular momentum of equal length $L = 1$ and oriented parallel to one another. The energy of these states is (nearly) linear in spin polarization ζ (and thus it remains finite at $\zeta = 0$) giving rise to a gapless and continuous density of states. On the other hand, the correlations between charged (particlelike) skyrmions are expected to be of Laughlin type,⁴ meaning that the skyrmion pair eigenstates with the smallest \mathcal{R} are maximally avoided.^{5,6,28-30} This can be rephrased in terms of an effective spatial isolation of skyrmions from one another, and the absence of high-energy skyrmion-skyrmion collisions⁶⁶⁻⁶⁸—even at their larger densities, beyond the melting point of the Wigner crystal.⁴⁴⁻⁴⁹

We also determine a criterion for the occurrence of skyrmions in a system of spin- $\frac{1}{2}$ particles half-filling a spin-degenerate shell (e.g., in a system of electrons or Laughlin QP's at $\nu = 1$ and an arbitrary layer width w). We find that the particle-particle interaction pseudopotential⁵ $V_{ee}(\mathcal{R})$ must (i) be strongly repulsive (superharmonic^{28,29}) at $\mathcal{R} = 0$ to cause decoupling of the many-body states that avoid having $\mathcal{R} = 0$ pairs from all other states [skyrmions are exact eigenstates¹⁹ of the ideal short-range repulsion with $V_{ee}(0) = \infty$], and (ii) decrease sufficiently quickly with increasing \mathcal{R} between $\mathcal{R} = 1$ and 3, $\mathcal{R} = 3$ and 5, etc. to make the skyrmion energy decrease with increasing the topological charge K (size) and, in particular, bring the skyrmion energy band below the energy of the spin-polarized QP state. For systems with broken particle (\uparrow)-hole (\downarrow) symmetry (e.g., at $\nu = \frac{1}{3}$), the latter condition must be rephrased in terms of the particle-hole pseudopotential V_{eh} which (iii) must increase monotonically as a function of wave vector k .

The above criterion allows prediction and explanation of the occurrence or absence of skyrmions (and the resulting type of dependence of spin polarization ζ of the 2DEG on density or magnetic field) at an arbitrary filling factor ν , layer width w , density profile across the layer $\varrho(z)$, etc.—based on the analysis of the involved e - e , e - h , QE-QE, QH-QH, or QE-QH interaction pseudopotentials. This criterion is somewhat analogous to the one for Laughlin correlations in a partially filled shell,^{6,28-30} which explained compressibility³¹ of the spin-polarized $\nu = \frac{4}{11}$ state ($\nu = \frac{1}{3}$ state of Laughlin QE's) and incompressibility³² of the partially unpolarized state at the same $\nu = \frac{4}{11}$ ($\nu = \frac{1}{3}$ state of QE's).

Although the model used as well as the qualitative physical picture obtained seem adequate for the actual 2D systems studied experimentally, at least at $\nu = 1$ (where the experiments are most accurate) some of the numerical values

poorly agree with experimental data. For example, the charge-excitation gaps (total energy of a noninteracting skyrmion-antiskyrmion pair; not shown) consistently overestimate the experimental values.¹³ This inconsistency is known from previous calculations using similar approximations,⁶³ and it occurs even for the simplest case of the pair of smallest ($K=0$) skyrmion and antiskyrmion, that is for the unbound reversed-spin electron and the hole. Therefore, we attribute this inconsistency to using inadequate self-energies for these two quasiparticles (due to neglecting the inter-LL and intersubband excitations, disorder, the tilt of the magnetic field, and possibly other omitted effects present in the experiment) rather than in inaccurate description of their mutual interactions, which are the main focus of the present study.

II. MODEL

The model used here is essentially that of Ref. 32, except that in the present calculation we do not include valence-band holes. A finite system of N electrons is confined on Haldane sphere²² of radius R . The radial magnetic field B is due to Dirac monopole, whose strength $2Q$ is defined in units of flux quantum ϕ_0 , so that $4\pi R^2 B = 2Q\phi_0$ and $R^2 = Q\lambda^2$. The single-electron states^{22,69,70} are the eigenstates of magnitude and projection of angular momentum (l and m) and of the spin projection (σ), and form g -fold ($g = 2l + 1$) degenerate LL's labeled by $n = l - Q = 0, 1, 2, \dots$

The cyclotron energy $\hbar\omega_c \propto B$ is assumed much larger than the Coulomb interaction energy $E_C = e^2/\lambda \propto \sqrt{B}$, so that the excitations between LL's can be neglected, and only a single, isolated n th LL need be considered. On the other hand, no assumption is made about the Zeeman spin splitting E_Z (except that $E_Z < E_C$), which in GaAs can be made arbitrarily small by applying hydrostatic pressure. The ratio $\eta = E_Z/E_C$ of energy scales associated with spin and charge excitations (within an isolated LL) is a small free parameter of the model.

Unless it is much larger than λ , finite layer width w enters the problem by only modifying³³ the quasi-2D interaction pseudopotential $V(\mathcal{R})$. In the lowest LL, the effects due to a finite w can be adequately modeled by merely reducing the Coulomb energy scale E_C by a factor $\xi_w < 1$ compared to the ideal $w=0$ case ($\xi_w \sim \frac{1}{2}$ in typical samples). However, the situation is very different in higher LL's, where even if w is sufficiently small that the intersubband mixing can be neglected, it affects not only the overall energy scale⁶³ but the shape of $V(\mathcal{R})$, and thus the electron correlations^{6,28-30} as well. To address this problem, we have included finite w in the calculation for higher LL's by using the quasi-2D Coulomb e - e potential $V_d(r) = e^2/\sqrt{r^2 + d^2}$, which is known^{33,68} to reproduce well the pseudopotential $V(\mathcal{R})$ for $w \approx 5d$. Here, w is the effective layer width obtained from fitting the actual lowest-subband density profile $\varrho(z)$ with $\cos^2(\pi z/w)$. Certainly, with only one parameter to describe $\varrho(z)$, this model may be quite inaccurate for some physical situations, such as for asymmetrically doped QW's or heterojunctions, or for the effects in tilted magnetic fields. However, it captures all the essential physics of a symmetric layer in a per-

pendicular magnetic field, only the layer width w must be distinguished from the well width. In typical GaAs wells w is larger than the well width by 3–3.5 nm due to a finite barrier height.

The Hamiltonian of interacting electrons confined to the n th LL can be written as

$$H = \sum c_{m_1\sigma}^\dagger c_{m_2\sigma'}^\dagger c_{m_3\sigma'} c_{m_4\sigma} \langle m_1 m_2 | V | m_3 m_4 \rangle, \quad (1)$$

where $c_{m\sigma}^\dagger$ and $c_{m\sigma}$ are the electron creation and annihilation operators, and the interaction matrix elements are calculated for the potential $V_d(r)$ and they are connected with the pseudopotential $V(\mathcal{R})$ through the Clebsch-Gordan coefficients. Hamiltonian H is diagonalized in the basis of N -electron Slater determinants

$$|m_1\sigma_1 \cdots m_N\sigma_N\rangle = c_{m_1\sigma_1}^\dagger \cdots c_{m_N\sigma_N}^\dagger |\text{vac}\rangle, \quad (2)$$

where $|\text{vac}\rangle$ stands for the vacuum state. While using basis (2) allows automatic resolution of two good many-body quantum numbers, projection of spin ($S_z = \sum \sigma_i$) and of angular momentum ($L_z = \sum m_i$), the length of spin (S) and of angular momentum (L) are resolved numerically in the diagonalization of each (S_z, L_z) Hilbert subspace. In addition to S and S_z we will also use the quantum number $K = \frac{1}{2}N' - S$, where $N' = \min(N, g - N)$, which measures the number of reversed spins relative to the maximally polarized state.

The results obtained on Haldane sphere can be easily converted to the planar geometry. The charged particles (or excitations) in a magnetic field move along closed, circular (cyclotron) orbits, which are similarly quantized in both planar and spherical geometry. To convert between the two geometries L and L_z of the sphere must be appropriately^{71,72} replaced by the total and center-of-mass angular momentum projections on the plane M and $M_{\text{c.m.}}$. On the other hand, the magnetic field does not affect the orbits of neutral particles (or excitations), which move along the straight lines on a plane and along the great circles on a sphere. The quantization of motion of these objects occurs only in a finite system (such as a sphere), due to the unphysical boundary conditions. In our calculation this quantization is an artifact of that disappears in the limit of an infinite sphere radius $R/\lambda \rightarrow 0$. To convert to the planar geometry, the discrete spherical angular momentum L must be replaced by the continuous longitudinal wave vector k defined by the following relation $L = kR = k\lambda\sqrt{Q}$.

III. INTEGRAL QUANTUM HALL REGIME (LOWEST LANDAU LEVEL)

A. Exactly filled level

Let us begin the discussion of spin excitations of the 2DEG in the integral quantum Hall regime with the energy spectrum at precisely $\nu = 1$. The single-spin-flip excitations [spin waves (SW)] at this filling factor were first identified by Kallin and Halperin,⁷³ and the excitation spectra for an arbitrary spin polarization were previously studied in some detail in the context of the finite-size quantum Hall

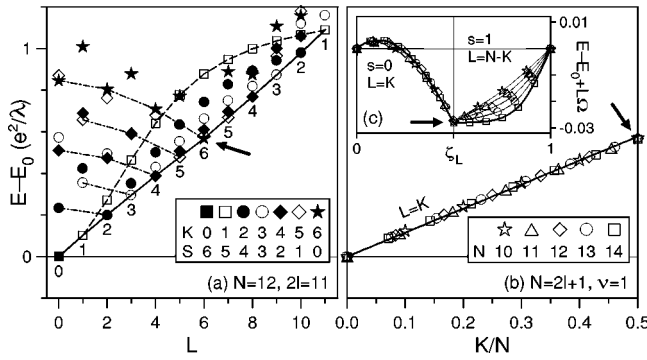


FIG. 1. (a) The energy spectrum (Coulomb energy E versus angular momentum L) of the system of $N=12$ electrons in the lowest ($n=0$) LL of degeneracy $g=2l+1=12$ calculated on the Haldane sphere. S is the total spin, and $K=\frac{1}{2}N-S$ is the number of reversed spins. Dashed line: single spin wave; dash-dotted lines: states containing equal numbers of $L=1$ spin waves; solid line: condensates of noninteracting $L=1$ spin waves. (b) The energy dispersion (energy E versus spin polarization $\zeta=K/N$) for the spin-wave condensate states at $\nu=1$, calculated for $N\leq 14$ electrons exactly filling their lowest LL ($N=2l+1$). (c) The dispersion of (b), but with inclusion of a lateral confinement term $L\Omega$ and plotted as a function of normalized angular momentum ζ_L . λ is the magnetic length and arrows mark the same state in each frame.

droplets.^{51,52} In this section we review some of their results as they are confirmed by our calculations for the purpose of making the comparison with the $\nu=\frac{1}{3}$ case more clear. Our original conclusions presented in this section are those regarding the SW-SW interaction and, in particular, the condensation of SW's into a state with finite angular momentum.

In Fig. 1(a) we show the finite-size spectrum for $N=12$ electrons on Haldane sphere with LL degeneracy $g=2l+1=N$. This and all other spectra in Secs. III–V have been calculated assuming zero width ($w=0$) of electron layer, while the spectra for $w>0$ will be discussed in Sec. VI. In all the spectra in the paper the excitation energy $E-E_0$ is given in the units of $E_C=e^2/\lambda$, measured from the energy E_0 of the lowest maximally spin-polarized state (here, the $\nu=1$ ground state), excludes the Zeeman energy E_Z , and is plotted as a function of total angular momentum L (for waves, related to longitudinal wave vector k by $L=kR=k\lambda\sqrt{Q}$). Different symbols mark multiplets of different total spin S and only the lowest state is shown at each L and S . The labels show the number K of reversed spins (relative to the maximally polarized state).

It is well known^{73,19} that (at least without disorder) even in the absence of the Zeeman energy gap ($E_Z=0$) the ground state (GS) of the 2DEG at $\nu=1$ is spin polarized (ferromagnetic) and translationally invariant (hence, nondegenerate). At the exact half filling of a spin-degenerate LL, this GS is the only state with zero projection onto the most strongly repulsive $e-e$ pair eigenstate at $\mathcal{R}=0$. Indeed, Fig. 1(a) shows a GS at $K=L=0$. Because of the complete filling of the spin- \downarrow $n=0$ LL in the ferromagnetic GS, the only excitations below the cyclotron gap $\hbar\omega_c$ are the spin waves (SW's) with $L=1,2,3,\dots$. A single SW consists of a vacancy (hole) in the spin- \downarrow level and an electron in the spin- \uparrow

level, and is also called a spin-exciton. Its dispersion curve is given by⁷³ $E_{SW}(k)=E_0+E_Z+E_C\sqrt{\pi/2}[1-\exp(-\kappa^2)I_0(\kappa^2)]$, where $\kappa=k\lambda/2$ and I_0 is the modified Bessel function, and it is shown in Fig. 1(a) with a dashed line.

Interestingly, Fig. 1(a) shows that a single SW is not the lowest-energy excitation at $L>1$. Instead, the lowest excitations form a band with $K=L$ and energy nearly linear in K (solid line). This (nearly) linear dependence $E(K)$ can be interpreted as the (nearly) perfect decoupling of SW's each with $L=1$, earlier pointed out by Oaknin *et al.*⁵¹ Denoting the energy of one SW with $L=1$ by ε_{SW} allows us to approximate the lowest state with K reversed spins by $E_{SW}(K)=E_0+K\varepsilon_{SW}$. Note that a pair of SW's each with $L=1$ can be in either parallel or antiparallel state, with $K=2$ and pair angular momentum $L=2$ or 0 , respectively. It follows from Fig. 1(a) that the $L=2$ state containing a pair of SW's is noninteracting, while the $L=0$ state of such a pair is repulsive (the SW-SW repulsion energy at $L=0$ decreases as a function of system size, and for $N=12$ it is $\approx 0.03E_C$). Similarly, the states containing the number K of SW's each with $L=1$ (see dash-dotted lines) can have total angular momentum $L=K, K-2, K-4, \dots$. However, only the $L=K$ state does not contain any SW-SW pairs with $L=0$, and thus only this state has energy $E=E_{SW}(K)$, while all others have $E>E_{SW}(K)$.

Although we have not calculated the actual overlaps, the above arguments lead us to the conjecture that the $L=K$ low-energy excitations of a 2DEG at $\nu=1$ are the uniquely ordered states of noninteracting SW's. In these states, the number K of SW's all have the same angular momentum of one (and the same energy ε_{SW}), and they are all oriented in parallel to give a sum of $L=K$. In the following these many-SW states will be denoted W_K and referred to as SW condensates.

The exact mapping⁵³ between the two-spin electron system ($\uparrow\downarrow$) and the electron–valence-hole ($e-h$) system in the lowest LL allows the expression of the above statement in terms of the interaction between interband magnetoexcitons; namely, the $L=1$ excitons in the lowest LL do not interact with one another and they condense into correlated $L=K$ states. Although similar, this symmetry is independent from the well-known “hidden symmetry”⁷⁴ of the $e-h$ systems. The latter is a direct consequence of the invariance of the interaction energy of the $\uparrow\downarrow$ systems under the rotations of the total spin. It is exact in the lowest LL and, among other effects, causes decoupling of $L=0$ excitons.^{53,66–68} In contrast, the symmetry presented here expresses the tendency for the system of many excitons (SW's) each with $L=1$ to avoid having exciton pairs in higher-energy eigenstates with $L=0$, and instead having $L=2$ for each pair. Since the energy of the $L=0$ pair is finite, the two-exciton angular momentum is not conserved in the interacting many-exciton system, and this symmetry is only approximate. It is somewhat analogous to the symmetry that describes the tendency of the electrons in the lowest LL to avoid the high-energy pair eigenstates and is responsible for Laughlin correlations.^{5,6,28–30}

To determine the energy spectrum of an infinite 2DEG we have compared data obtained for different electron numbers

$N \leq 14$. As shown in Fig. 1(b), it turns out that the energy E_W of W_K excitations is a nearly linear function of the “relative” spin polarization $\zeta = K/N$. Namely, $E_W = E_0 + u\zeta$, with the slope that we estimate as $u \approx 1.15E_C$. This fact reflects the extended character of the SW’s and has a couple of obvious consequences for the thermodynamic limit of $N \rightarrow \infty$: (i) For any $E_Z \neq 0$, the interaction energy $E_W(K) - E_0 \propto K/N$ of each correlated W_K state is negligible compared to its total Zeeman energy, KE_Z . (ii) The gap for spin excitations is precisely the Zeeman gap E_Z ; if this gap can be closed (e.g., by applying hydrostatic pressure), the $\nu = 1$ ferromagnetic GS becomes gapless, the density of states for the W_K excitations becomes continuous, and a macroscopic number $\sim Nk_B T/u$ of noninteracting SW’s occur at an arbitrarily small finite temperature.

To remove the dominant linear term from $E_W(\zeta)$ and study the small nonlinear correction, in Fig. 1(c) we plot the energy spectrum shifted by an additional linear term $L\Omega$. Physically, this term describes a harmonic lateral confinement applied to a finite-size planar $\nu = 1$ droplet.^{50–52} The confinement strength Ω is chosen so that the $\nu = 1$ GS is degenerate with the next “compact droplet” state at $K = 1$ and $L = N - 1$ (compact-droplet eigenstates^{50–52} C_K with $K \geq 1$ are created by $\prod_{i=0}^{K-1} c_{l-i, \uparrow}^\dagger c_{-l+i, \downarrow}$ acting on $C_0 = \prod_{m=-l}^l c_{m, \downarrow}^\dagger |\text{vac}\rangle$, i.e., on the $\nu = 1$ GS). To compare data for different N , the energy is plotted as a function of normalized angular momentum $\zeta_L = (L - s^2)/(N - 2s)$, where s is equal to K of the nearest C_K state, so that ζ_L increases with L and equals K in each C_K state (note that $\zeta_L = \zeta$ for $\zeta_L \leq \frac{1}{2}$).

Clearly, the nonlinear correction to $E_W - E_0$ is also a well-defined function of ζ or ζ_L with a minimum at $\zeta_L = \frac{1}{2}$, i.e., in the highly correlated singlet ($S = 0$) state marked with arrows. If both E_Z and N are sufficiently small, the energy of this singlet state, $E_W(\frac{1}{2}) - E_0 + \frac{1}{2}N\Omega \approx -0.03E_C + \frac{1}{2}NE_Z$ can remain negative. Then, the sublinear behavior of $E(\zeta_L)$ between $\zeta_L = 0$ and $\frac{1}{2}$ implies an abrupt transition from the $\nu = 1$ ferromagnetic GS to the $\zeta_L = \frac{1}{2}$ singlet GS (skipping intermediate spins S) of a finite-size quantum Hall droplet as a function of Ω .^{51,52} On the other hand, the superlinear behavior of $E(\zeta_L)$ between $\zeta_L = \frac{1}{2}$ and 1 implies that a further decrease of Ω drives a quantum Hall droplet from the $S = 0$ state to the $S = N - 1$ compact state through a series of GS’s with all intermediate spins.

B. Additional particle or hole in filled level

What truly ignites the abrupt depolarization of a finite $\nu = 1$ droplet under the variation of the lateral confinement is the insertion of an additional reversed-spin particle (from the edge) into the bulk of the droplet. This effect is a consequence of a more general phenomenon—the occurrence of particlelike charged excitations with macroscopic spin called skyrmions.^{15–19} Let us consider a (possibly infinite) system of spin- $\frac{1}{2}$ fermions half filling ($\nu = 1$) a degenerate ($E_Z = 0$) shell of single-particle states and forming a ferromagnetic GS in accordance with a standard atomic Hund’s rule. Depending on details of the interaction pseudopotential, ad-

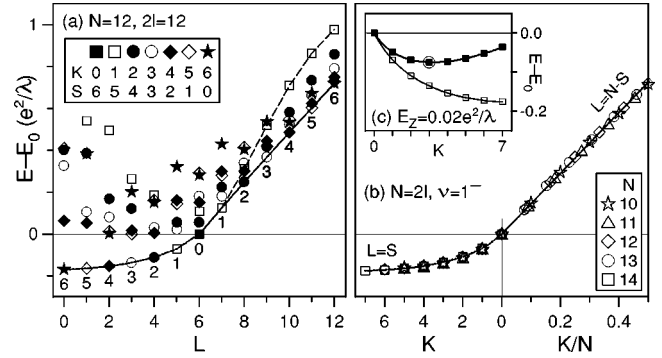


FIG. 2. (a) Same as Fig. 1(a) but for $g = 2l + 1 = 13$. Dashed line: single spin wave in the presence of a hole; solid line: spin-wave condensate in the presence of a hole; dotted line: skyrmions. (b) Same as Fig. 1(b) but for $N = 2l$, i.e., one hole in the $\nu = 1$ state. The skyrmion and spin-wave condensate energies are plotted as a function of numbered of reversed spins K and spin polarization $\zeta = K/N$, respectively. (c) Open and full symbols: skyrmion dispersion of (b) for $N = 14$ without and with Zeeman energy KE_Z .

dition of a single particle or hole to such system may or may not cause its complete depolarization, that is a transition to a highly correlated singlet GS in which half of the electrons flipped their spins. Examples of systems in which such depolarization does or does not occur are electrons in the lowest LL or in an atomic shell, respectively.

The effect of removing an electron from a $\nu = 1$ GS is presented in Fig. 2(a), showing the energy spectrum of $N = 12$ electrons at $2l = 12$ (i.e., N smaller by one than the LL degeneracy $g = 2l + 1$). Similar spectra were first analyzed by Xie and He,³⁷ and again, we are repeating some of their conclusions here to make the comparison with the $\nu = \frac{1}{3}$ more clear. Because of the exact particle-hole symmetry in the lowest LL, the system at $N + 1 = g$ can be viewed as containing either one hole or one extra electron in a $\nu = 1$ GS of 13 electrons, and will be labeled as $\nu = 1^\pm$. Clearly, while the right-hand part of the spectrum (the single SW and $L = N - S$ bands at $L \geq \frac{1}{2}N$) resembles that of Fig. 1(a), a new band of states with $L = S$ appears at $0 \leq L < \frac{1}{2}N$, at an energy below the lowest $K = 0$ state. These are the skyrmion states of topological charge K , denoted here by S_K and first identified in nuclear physics by Skyrme¹⁵ and in the fractional quantum Hall systems by Sondhi *et al.*¹⁹ In these states, $K = 1, 2, \dots$, SW’s bind to a hole (to be more accurate, these are antiskyrmion states; skyrmions are their particle-hole conjugate states consisting of SW’s bound to an electron). In the e - h picture, the S_K states map⁵³ onto the charged multiexcitons^{54,55} X_K^\pm , consisting of K e - h pairs bound to an extra e or h .

To understand the energy spectrum of an infinite 2DEG at $\nu = 1^\pm$, in Fig. 2(b) we compare data obtained for different N . Similarly to the $L = K$ band at $\nu = 1$, the energy of $L = N - S$ states at $\nu = 1^\pm$ turns out to be a nearly linear function of the spin polarization $E'_W = E_0 + u'\zeta$, only with a slightly increased slope, $u' = u + v$ with $v \approx 0.3E_C$. Since the angular momentum $L = N - S$ can be obtained by adding $L = K$ for the W_K condensate and $l = \frac{1}{2}N$ for the single hole (assuming their parallel orientation), the positive energy ν/N

can be interpreted as the repulsion between a finite-size hole and (oriented “parallel” to it) one extended SW.

In view of their charged multiexciton interpretation⁵⁵ in the e - h picture,⁵³ it is not surprising that the relevant quantum number to label the $L=S$ skyrmion excitations S_K in an infinite system is K (and not $\zeta=K/N$ appropriate for W_K condensates). Indeed, Fig. 2(b) shows that the excitation energy $E_S - E_0$ of the S_K states is a function of K (rather than of ζ), with the discrete series of values of $E_S(K)$ quickly converging as N is increased [careful extrapolation to $N \rightarrow \infty$ gives⁵⁵

$$E_S(K) - E_0 = -0.0529, -0.0828, -0.1018, \dots, \\ -\frac{1}{4}\sqrt{\pi/2} = -0.3133 \text{ for } K=1, 2, 3, \dots, \infty;$$

all in units of E_C]. The fact that $E_S(K) - E_0 < 0$ means that in the S_K state the attraction between the hole (or electron) and K SW's overcomes the creation energy of these SW's, and the ferromagnetic state with $K=0$ may become unstable. Most importantly, as shown in Fig. 2(c), if E_Z is nonzero but smaller than $E_0 - E_S(K)$, then regardless of its actual value a particle-like GS will occur with an excitation gap that is much smaller than the gap at precisely $\nu=1$ (which is E_Z). In other words, introduction of additional electrons (or holes) to the incompressible $\nu=1$ GS with a gap E_Z will cause significant reduction of the gap for spin excitations,³⁶ and the objects that are able to reverse spin at low energy (much below E_Z) are finite-size charged particles (skyrmions) that move in the underlying $\nu=1$ fluid on electronlike cyclotron orbits. The ability of (mobile) skyrmions to increase and decrease spin at an energy cost that is small compared to and largely independent of E_Z (all in contrast to the $\nu=1$ state) was first pointed out by Fertig *et al.*³⁶ Among other consequences, it causes critical magnetic field dependence of the spin relaxation rate for magnetic particles interacting with the 2DEG, such as ions, nuclei, or charged excitons.⁷⁵

While the decoupling of SW's follows from the linear dependence of E_W on K (or E'_W on K , for the SW's in the presence of an extra electron or a hole), the nature of interaction between skyrmions follows from an earlier study of the e - h complexes. It was shown^{66,67} that the interaction between any pair of charged excitons X_K^\pm (X_0^\pm means an electron or a hole) with equal charge ($\pm e$) but possibly different sizes ($K \neq K'$) is repulsive and similar to the e - e interaction. In particular, all K - K' repulsion pseudopotentials (defined⁵ as the pair interaction energy V as a function of relative pair angular momentum \mathcal{R}) have short range, implying^{28,29} that an X_K^\pm particle will have Laughlin correlations⁴ with all other $X_{K'}^\pm$ particles in the system. The appropriate definition of the short-range repulsion is that V increases more quickly than linearly as a function of average pair separation $\sqrt{\langle r^2 \rangle}$, when \mathcal{R} is increased,²⁸⁻³⁰ and Laughlin correlations are described by an appropriate Jastrow prefactor in the many-body wave function and mean the tendency to avoid the pair states with maximum repulsion (minimum average separation). This implies that (at low temperature) an X_K^- does not undergo high-energy collisions with any other $X_{K'}^-$ charges. Since the same must hold for skyrmions, we conclude that regardless of their

size (K) or density, the skyrmions (at sufficiently low density, i.e., at ν sufficiently close to $\nu=1$) will be effectively isolated from one another. This makes finite size skyrmions well-defined quasiparticles, virtually unperturbed by the skyrmion-skyrmion scattering, and excludes many-skyrmion effects from a possible spin coupling of a 2DEG to magnetic particles.

A consequence of Laughlin skyrmion-skyrmion correlations with possible experimental implications is the following dependence of the average skyrmion size $\langle K \rangle$ and mobility in a macroscopic system on the filling factor. At ν sufficiently close to 1, $\langle K \rangle$ remains independent of ν and equal to the value K describing an isolated skyrmion (a function of $\eta=E_Z/E_C$ and w ; see Figs. 11 and 12). In this “dilute” regime, an increase (or decrease) of ν away from 1 causes the increase of the effective filling factor of skyrmions (or antiskyrmions) $\nu_{\text{sky}}=K|1-\nu|$ from 0 to 1, without distortion of their individual wave functions. Actually, it is well known from earlier, field-theoretical studies using the $NL\sigma$ model approach¹⁹⁻²¹ that at sufficiently small ν_{sky} skyrmions freeze into a Wigner crystal.⁴⁴⁻⁴⁹ This crystal is also known^{47,48} to melt at a critical value of ν_{sky} , and we only notice here that the fluid phase will have Laughlin correlations.

When ν reaches a critical value of $1 \pm \delta(\eta, w)$, corresponding to $\nu_{\text{sky}}=1$, the crossover to the incompressible regime takes place. In this regime, skyrmions remain locked in a rigid (fluid) state and uniformly cover the entire 2DEG area. A further change of ν beyond $1 \pm \delta$ causes their compression, that is a (linear) decrease of $\langle K \rangle$, but it does not affect the complete coverage. Clearly, any experimentally observed feature sensitive to the individual skyrmion wave function will remain constant at $|1-\nu| < \delta$ and depend on ν outside this range. On the other hand, the features that depend on the 2DEG coverage might be more sensitive to ν in the dilute regime.

IV. FRACTIONAL QUANTUM HALL REGIME

Let us now, following Kamilla *et al.*⁵⁶ and MacDonald and Palacios,⁵⁹ turn to the question of whether the spin excitations analogous to the spin-wave condensates and S_K skyrmion particles described in the preceding section might also occur in the fractional quantum Hall regime. On one hand, it is known that Laughlin correlations⁴ in an electron system near $\nu=(2p+1)^{-1}$ (where p is an integer) allow the mapping²³⁻²⁵ of the low-energy states onto the noninteracting CF states with an effective filling factor $\nu^* \approx 1$. This mapping is done by replacing the electron LL degeneracy g by $g^*=g-2p(N-1)$, which can be interpreted as attachment of $2p$ magnetic flux quanta to each electron. On a sphere,⁶ this replaces $2l=2Q \approx (2p+1)(N-1)$ by $2l^*=2Q^* \approx N-1$.

On the other hand, it is the specific form of the interactions between the reversed-spin electrons and holes at $\nu=1$ that causes occurrence of W_K and S_K excitations, and the interaction between these excitations in an electron system is quite different from the residual interaction between CF's. A well-known example demonstrating that the analogy be-

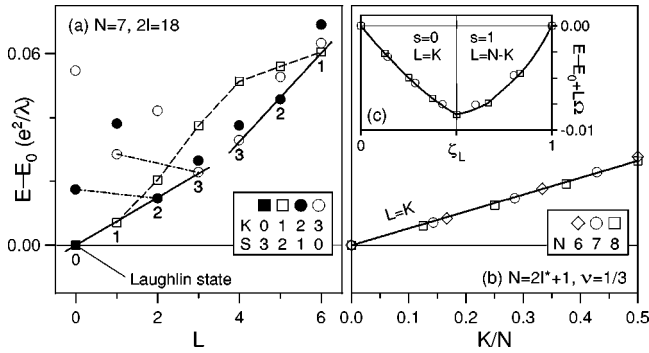


FIG. 3. Same as Fig. 1 but for the LL degeneracy $g=2l+1$ corresponding to $\nu=\frac{1}{3}$ (completely filled CF LL).

tween the electron and CF systems sometimes fails because of different interactions is the postulate of similar Laughlin correlations at the $\nu=(2p+1)^{-1}$ fillings of electron or CF LL's, giving rise to Haldane hierarchy of incompressible fractional quantum Hall states.²² For example, the $\nu=\frac{1}{3}$ states of the vacancies in the spin- \downarrow $n=0$ CF LL (Laughlin QH's) and of particles in the spin- \downarrow $n=1$ CF LL (Laughlin QE's) or in the spin- \uparrow $n=0$ CF LL (reversed-spin quasielectrons^{76,77} QE_R) correspond to the polarized strongly incompressible $\nu=\frac{2}{3}$ and compressible $\nu=\frac{4}{11}$ states,³¹ and to the partially unpolarized weakly incompressible³² $\nu=\frac{4}{11}$ state, respectively.

The examples of energy spectra at $\nu\approx\frac{1}{3}$ are shown in Figs. 3(a), 4(a), and 5(a). The values of N and $2l$ are chosen so that $g^*=7$ in each frame and the "reference" state with $K=0$ and $E=E_0$ is the Laughlin state ($g^*=N$ and $\nu^*=1$), one QH ($g^*=N+1$ and $\nu^*=1^-$), and one QE_R ($g^*=N-1$ and $\nu^*=1^+$) in Figs. 3, 4, and 5, respectively. Some of the energies for smaller values of N and/or K have been recently obtained by MacDonald and Palacios.⁵⁹

Clearly, the SW dispersion $E_{SW}(K)$, the linear $E_W(K)$ band, as well as the $E_S(K)<0$ band are all present in the spectra, in analogy to Figs. 1(a) and 2(a). What is visibly different from the $\nu=1$ spectra is a smaller energy scale (predominantly due to a fractional charge of involved QH, QE, and QE_R quasiparticles), and a discrepancy between the

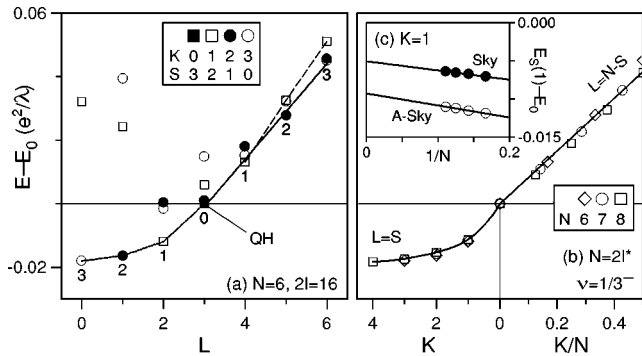


FIG. 4. Same as Fig. 2 but for the LL degeneracy $g=2l+1$ larger by one than the value at $\nu=\frac{1}{3}$ (one hole in CF LL). Inset: Energies $E_S(1)$ of skyrmions (data in following Fig. 5) and antiskyrmions (present figure) with $K=1$ as a function of inverse electron number, N^{-1} .

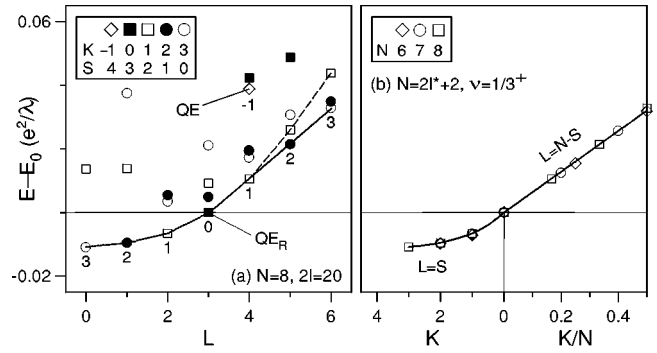


FIG. 5. Same as Fig. 2 but for the LL degeneracy $g=2l+1$ smaller by one than the value at $\nu=\frac{1}{3}$ (one reversed-spin particle in CF LL).

$\nu=\frac{1}{3}^-$ and $\frac{1}{3}^+$ spectra. The latter reveals an only approximate particle-hole symmetry within the lowest CF LL (it was shown earlier³² that the QH-QH and QE_R - QE_R interactions are quite different). This implies broken symmetry between the skyrmion and antiskyrmion states,⁵⁹ in contrast to the $\nu=1$ case.

Let us analyze the spectra in Figs. 3, 4, and 5 in more detail. The charge excitations of the Laughlin $\nu=\frac{1}{3}$ GS in Fig. 3(a) are all above the magnetoroton gap ($E-E_0\approx 0.081E_C$ for $N=7$) and are not shown. The spin excitations include a single SW (an excitonic bound state of a QH- QE_R pair) marked by a dashed line. Below a single SW, there is a linear band of W_K states with $L=K$ which, in analogy to the $\nu=1$ spectra, can be interpreted as the condensates of K noninteracting SW's each with $L=1$ and all having parallel angular momenta. As shown in Fig. 3(a), similarly as at $\nu=1$, the energy of W_K states is a nearly linear function of ζ , $E_W=E_0+u\zeta$, with $u=0.053E_C$ (much less than $\frac{1}{9}$ times the value for $\nu=1$, that would only take into account the smaller charge of $\pm\frac{1}{3}e$ for QH's and QE_R 's). As at $\nu=1$, $E_W(K)-E_0\propto K/N$ means no gap and a continuous density of states if the Zeeman gap can be closed. The comparison of Figs. 3(c) and 1(c) shows that applying lateral confinement can also force the spin-polarized finite-size $\nu=\frac{1}{3}$ fractional quantum Hall droplet to undergo transition to the spin-singlet ($S=0$) correlated GS at $L=\frac{1}{2}N$, just as it was at $\nu=1$.

The linear bands at $L=N-S$ found in Fig. 2 occur also at $\nu=\frac{1}{3}^\pm$ in Figs. 4 and 5. By analogy, these states correspond to a number K of SW's each with $L=1$, coherently created in the presence of a QH or QE_R . The QH-SW and QE_R -SW interaction constants v , obtained from the $E'_W=E_0+(u+v)\zeta$ fits as shown in Figs. 4(b) and 5(b) are remarkably different, $0.030E_C$ and $0.011E_C$, respectively.

The discrete skyrmion and antiskyrmion bands at $L=S$ in Figs. 4 and 5 also resemble their $\nu=1$ counterpart in Fig. 1, and the energies $E_S(K)$ all seem to converge when $N\rightarrow\infty$. For example, in Fig. 4(c) the linear extrapolation of the energies of skyrmions (Sky) and antiskyrmions (A-Sky) with $K=1$, obtained for $N\leq 9$, gives $E_S(1)-E_0=-0.0050E_C$ and $-0.0093E_C$, respectively. These are the critical values of the Zeeman energy E_Z , below which these excitations can be observed experimentally. Note also that the $K=1$ skyr-

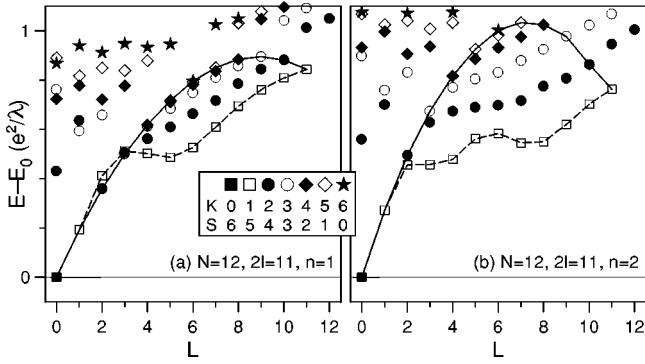


FIG. 6. Same as Fig. 1(a) but for electrons confined to the first (a) and second (b) excited LL.

mion and antiskyrmion states are perfect analogs of the interband charged exciton states, except that e and h are replaced by QE_R and QH . In analogy to $\nu=1$, the values of $E_S(1)-E_0$ measure the binding energies of such (fractionally) charged CF spin excitons $X_{CF}^- = 2QE_R + QH$ and $X_{CF}^+ = 2QH + QE_R$.

It might seem surprising that the spin excitations at $\nu=1$ and $\frac{1}{3}$ are similar despite different interactions between electrons and CF's. However, of all three types of QP's at $\nu=\frac{1}{3}$, only QH and QE_R are involved in the low-energy spin excitations W_K and S_K , and the QH - QH , QH - QE_R , and QE_R - QE_R pseudopotentials describing their interactions,^{31,32} are all quite similar to the e - e and e - h pseudopotentials in the lowest electron LL. On the other hand, the QE 's, whose interaction (at short range) with one another and with other QP's is very different,³¹ do not participate in W_K and S_K excitations.

V. INTEGRAL QUANTUM HALL REGIME (HIGHER LANDAU LEVELS)

Another system with an identical structure of the single-particle Hilbert space but with different interactions is a nearly completely filled higher ($n>0$) LL, experimentally realized in the 2DEG at $\nu \approx 2n+1$. Note that if skyrmions would indeed occur at $\nu=3,5$, etc., they should be observed even more easily than at $\nu=1$ because of the weaker magnetic field B (at the same 2DEG density), and thus smaller $\eta = E_Z/E_C \propto \sqrt{B}$.

In this section we shall discuss the results for an ideal 2D system with zero layer width $w=0$. The energy spectra analogous to those in Figs. 1(a) and 2(a) but calculated for $n=1$ and 2 are shown in Figs. 6 and 7. The same the number of electrons $N=12$ (in the n th LL; the lower LL's are completely filled) and the angular momenta $l=11$ and 12 have been chosen, yielding the monopole strength $2Q=2(l-n)$. Clearly, none of the abovediscussed features of the $\nu=1$ or $\nu=\frac{1}{3}$ ($\nu^*=1$) spectra are present at $\nu=3$ or 5.

Let us begin with Fig. 6 for N equal to the LL degeneracy, g . A single SW (dashed lines; for dispersion see Ref. 73) is generally the lowest-energy spin excitation (at any L), and the W_K bands (identified by comparison of the pair-correlation functions) have higher energy and are no longer

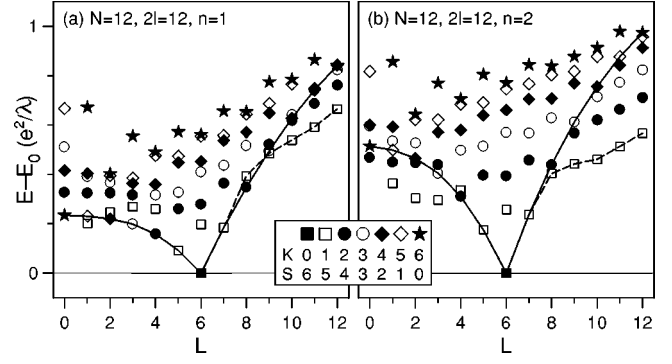


FIG. 7. Same as Fig. 2(a) but for electrons confined to the first (a) and second (b) excited LL.

linear. The sublinear and nearly parabolic $E_W \approx E_0 + u\zeta + y\zeta^2$ can be interpreted as an attraction between the $L=1$ SW's and, based on data for $N \leq 14$, we find $u=2.41E_C$ and $y=-1.62E_C$ for $n=1$, and $u=3.45E_C$ and $y=-2.88E_C$ for $n=2$. In a finite-size $\nu=3$ or 5 droplet, as a result of this attraction, the spin-singlet condensate of $K=\frac{1}{2}N$ SW's (marked with arrows in Fig. 1) is an excited state at any strength of confinement (Ω), and the edge reconstruction of the $\nu=3$ or 5 ferromagnetic GS (C_0) occurs directly to the next compact-droplet state C_1 . This different behavior might be probed in a transport experiment by sending a reversed-spin electron over a quantum dot containing a compact droplet. It seems that a reversed-spin carrier would induce and bind SW's when sent through a $\nu=1$ or $\frac{1}{3}$ state, and travel ballistically for $\nu=3$ or 5.

The lack of response to an addition of a reversed-spin electron (from the edge into the inside of the droplet) must mean unbinding of skyrmions at $\nu=3$ or 5. Indeed, the S_K states in the spectra for $N+1=g$ in Fig. 7 all have $E>E_0$. This means no skyrmions in higher LL's at any value of E_Z . In contrast to the situation near $\nu=1$ or $\frac{1}{3}$, the GS both precisely at $\nu=3$ or 5 and in the vicinity of these values remains maximally spin polarized even in the absence of Zeeman splitting. In the e - h picture, the result is that no bound charged-exciton states X_K^- occur in higher LL's (in the absence of inter-LL mixing and finite well width effects).

VI. EFFECTS OF FINITE LAYER WIDTH

As was first predicted by Cooper⁶³ and later confirmed experimentally by Song *et al.*,⁶⁴ skyrmions become the lowest-energy charged excitations in higher LL's as well, if only the layer width w is sufficiently large. We have calculated the spin-excitation spectra analogous to those of Figs. 6 and 7 but for $w=3\lambda$, and show them in Figs. 8 and 9. For the exact fillings of the $n=1$ and 2 LL's ($\nu=3$ and 5; Fig. 8), the $E_W(K)$ energy bands which were strongly sublinear for $w=0$ now become nearly linear (similar to the lowest LL; see Fig. 1). This indicates vanishing of the SW-SW attraction, and reoccurrence of the condensate of ordered $L=1$ SW's. For an additional hole in the $n=1$ and 2 LL's ($\nu=3^-$ and 5^- ; Fig. 9), the skyrmion energy bands which had $E_S(K)>E_0$ for $w=0$ now have $E_S(K)<E_0$ (again, similar

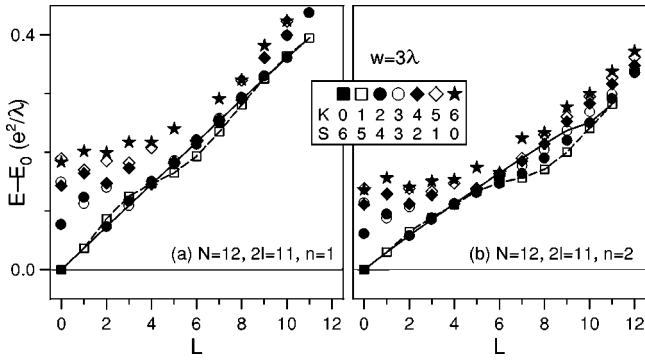


FIG. 8. Same as Fig. 6 but for a finite width $w=3\lambda$ of a quasi-2D electron layer.

to the lowest LL; see Fig. 2). This indicates stability of skyrmions in higher LL's in a wide quasi-2D layer (at sufficiently small E_Z). Also in Fig. 9, the $L=N-S$ bands of states (containing K SW's created coherently in the presence of a hole) become now nearly linear, in contrast to the behavior in Fig. 7 but similarly to Fig. 2.

In Fig. 10 we compare the energies of skyrmions with $K=1, 2$, and $\frac{1}{2}N$ (the latter state has $S=0$ and would correspond to an infinite-size skyrmion in the $N=\infty$ limit) plotted as a function of the layer width w . Clearly, the skyrmion energy is more sensitive to w in higher LL's. The "binding energies" $E_S(K) - E_0$ that were all positive for $w=0$ in Fig. 7 change sign at $w/\lambda=2$ to 3, depending on K and n . Note that our critical values of w are considerably higher than those predicted by Cooper. For example, for the $n=1$ LL, his critical parameter $a=0.09\lambda$ for the Gaussian density profile, $\rho(z) \propto \exp(-z^2/2a^2)$ corresponds to $w \approx 0.5\lambda$ for our $\rho(z) \propto \cos^2(z\pi/w)$. This discrepancy indicates slow convergence of the energy of an infinite ($S=0$) skyrmion with the electron number N . However, our critical values are certainly more appropriate for small skyrmions which are the ones that might be observed experimentally. Let us compare these values with a pair of experiments in which the skyrmions were and were not observed at $\nu=3$. Taking parameters after Song *et al.*,⁶⁴ who observed skyrmions with $K \leq 2$ ($B=2.15$ T and well width of 30 nm yielding $w=33.5$ nm) gives $w/\lambda=1.9$, just above our critical value (see Fig. 12 for data extrapolated to $N \rightarrow \infty$; the experimental widths have been

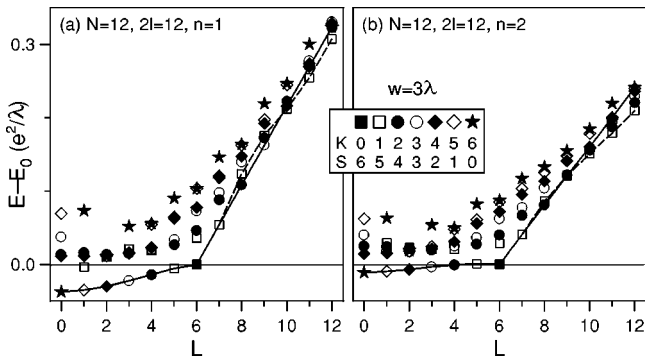


FIG. 9. Same as Fig. 7 but for a finite width $w=3\lambda$ of a quasi-2D electron layer.

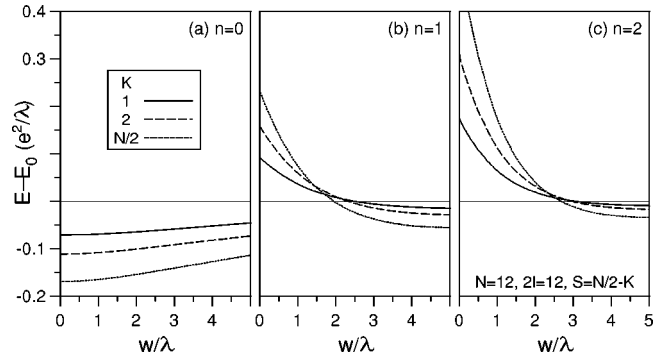


FIG. 10. The energy E of skyrmions with $K=1, 2$, and $\frac{1}{2}N$ as a function of the layer width w , calculated on Haldane sphere for $N=12$ electrons in the lowest (a), first excited (b), and second excited (c) LL. λ is the magnetic length.

marked with arrows). On the other hand, taking the parameters after Schmeller *et al.*,⁶² who did not observe skyrmions ($B=2.3$ T and well widths of 14 and 20 nm yielding $w=17.5$ and 23.5 nm) gives $w/\lambda=1.03$ and 1.40 for their two samples, both below our critical value but above that of Cooper.

Using the data of Fig. 10 one can calculate a phase diagram for the occurrence of skyrmions with a given number of reversed spins K as a function of the layer width w and the Zeeman energy E_Z . Such a diagram is presented in Fig. 11 for the integral filling of the lowest and first two excited LL's ($\nu=1, 3$, and 5, respectively). Disorder, screening, or tilt of the magnetic field are all ignored in this diagram, although they can play a role in experimental conditions. To have a more reliable estimate of the critical w/λ and η for the occurrence of skyrmions with any $K \leq 1$, we have recalculated the curves for $K=1$ and 2 for much larger N (up to 50) and then, thanks to their regular dependence on N , were able to extrapolate them to the $N \rightarrow \infty$ limit. The resulting phase diagram, shown in Fig. 12, describes an infinite planar system, and it is consistent with the skyrmion energies reported by Palacios *et al.*⁵⁵ for $n=0$ and $w=0$. Remarkably, the critical value of E_Z for the lowest LL is quite insensitive to w over a wide range of layer widths. This is in contrast to the situation in higher LL's, for which the phase diagrams in Figs. 12(b), 12(c) show a similar fast increase of the critical E_Z with increasing w . The critical layer widths in the limit of vanishing Zeeman energy are $w/\lambda=1.8$ and 2.3 for $n=1$ and 2, respectively. In view of a recent study⁴³ that showed that the LL mixing only weakly affects the skyrmion energies in the layers of nonzero width, we expect our phase diagrams in Fig. 12 to be quite adequate for realistic experimental systems.

Finally, in Fig. 13 we present an analogous phase diagram for the $\nu=\frac{1}{3}$ fractional quantum Hall state. Due to the broken QE_R -QH symmetry, the diagrams for skyrmions (at $\nu=\frac{1}{3}^+$) and antiskyrmions (at $\nu=\frac{1}{3}^-$) are different, and they are both shown. The solid lines and shaded areas give the result for small systems: $N=7$ and $2l=19$ in frame (a), and $N=8$ and $2l=20$ in frame (b). The dashed lines give the critical values of E_Z at $w=0$ for the X_{CF}^+ , QE_R , and X_{CF}^- states, obtained from extrapolation of data for $N \leq 9$ to $N \rightarrow \infty$.

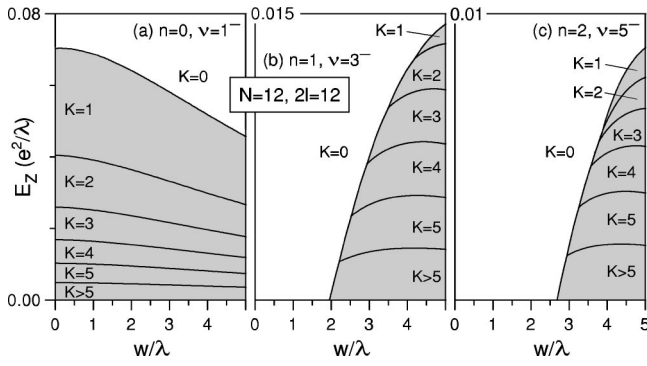


FIG. 11. Phase diagrams for the occurrence of skyrmions with K reversed spins in a quasi-2D electron gas of finite width w , calculated in the system of $N=12$ electrons in the lowest (a), first excited (b), and second excited (c) LL of degeneracy $g=2l+1=12$. E_Z is the Zeeman energy and λ is the magnetic length. The numbers in the top-left corners of frames (b) and (c) give the upper bounds of their vertical axes (the lower bound are zero in all frames).

VII. CRITERION FOR SKYRMIONS IN HALF-FILLED SPIN-DEGENERATE SHELL

Since (i) the finite width w enters the Hamiltonian (1) only through the pseudopotential $V(\mathcal{R})$, (ii) only a few leading parameters $V(0), V(1), V(2), \dots$, that correspond to a short average $e-e$ distance $\sqrt{\langle r^2 \rangle}$ significantly depend on w , and (iii) for $w=0$, the opposite behavior of $E_S(K) - E_0$ for the lowest and higher LL's results precisely from the enhanced value of $V(2)$ for $n \geq 1$, it is enough to study the dependence of the short-range part of $V(\mathcal{R})$ on w to understand the reoccurrence of skyrmions for $n \geq 1$ at $w \sim 2\lambda$. This dependence is illustrated in Figs. 14(a)–14(c). Clearly, increasing d (i.e., $w=5d$) suppresses more strongly the pseudopotential parameters at the even values of \mathcal{R} (open circles) corresponding to zero pair spin, specially the highly repulsive ones at $\mathcal{R}=0$ and 2 for $n=1$ and $\mathcal{R}=0, 2$, and 4 for $n=2$. While in any LL the strong suppression of $V(0)$ will eventually (at very large w) destroy skyrmions (all having no pairs with $\mathcal{R}=0$), there is a wide range of w in which skyrmions become stable also for $n > 0$.

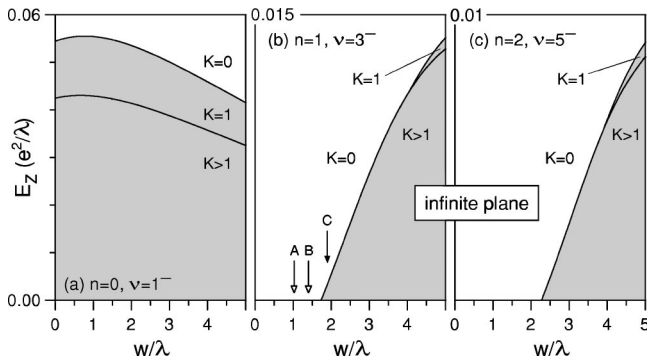


FIG. 12. Same as Fig. 11 but for an infinite planar system (critical values of Zeeman energy E_Z for each layer width w were obtained from extrapolation of data for electron numbers $N \leq 50$). In frame (b), arrows indicate the widths corresponding to experiments of Ref. 62 (a,b) and Ref. 64 (c).

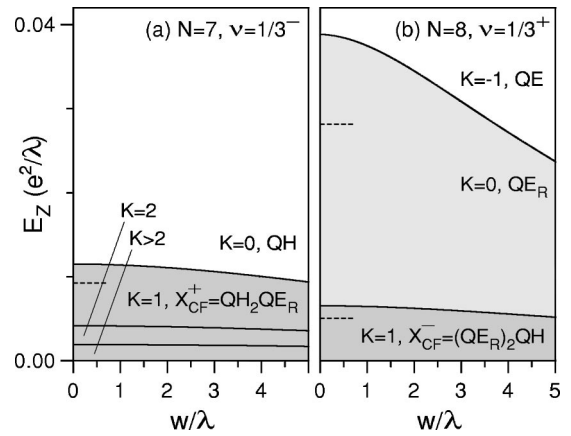


FIG. 13. Same as Fig. 11 but for the fractional quantum Hall states near $\nu = \frac{1}{3}$. Horizontal dashed lines mark the critical values of E_Z at $w=0$ obtained from extrapolation of the finite-size data for $N \leq 9$ to the $N \rightarrow \infty$ limit.

By comparing the values of w and d in Figs. 10(a)–10(c) and 14(a)–14(c), we find the following general conditions for the occurrence of skyrmions in a system of interacting spin- $\frac{1}{2}$ fermions half-filling a spin-degenerate shell: (i) $V(0)$ must be sufficiently large to cause decoupling of the many-body states without the $\mathcal{R}=0$ pairs (skyrmions) from all other many-body states and (ii) $V(\mathcal{R})$ must decrease with increasing $\mathcal{R} \geq 1$. Note that the latter condition (ii) is not immediately applicable to shells with broken particle (\uparrow)-hole (\downarrow) symmetry. Examples of such systems include electrons at $\nu = \frac{1}{3}$ (i.e., CF's at $\nu^* = 1$) but also the case when the pair of spin-degenerate LL's has different orbital indices n (which can be realized by making the Zeeman gap E_Z equal to the cyclotron gap $\hbar\omega_c$ in a magnetic material).⁷⁸ In these systems, the interaction Hamiltonian (1) is determined by a pair of (different) particle-particle and hole-hole pseudopotentials $V_{\uparrow\uparrow}(\mathcal{R})$ and $V_{\downarrow\downarrow}(\mathcal{R})$, where \mathcal{R} is an odd number (as required for two identical fermions), and a particle-hole continuous dispersion $V_{\uparrow\downarrow}(k)$, where k is the pair wave vector (on a sphere, $kR=L$). To rephrase condition (ii) in terms of $V_{\uparrow\downarrow}(k)$, we notice in Fig. 14 that the suppression of the maxima at $\mathcal{R}=2$ (for $n=1$) and $\mathcal{R}=2$ and 4 (for $n=2$) coincides with the disappearance of the corresponding roton minima in the spin-wave dispersion $V_{eh}(k)$, at $k\lambda \approx 2.1$ (for $n=1$) and $k\lambda \approx 1.5$ and 3.2 (for $n=2$).

A continuous evolution of the skyrmion energy spectrum $E_S(K)$ from the positive values (as for $n \geq 1$ and small w) to the negative values (as for $n=0$ or $n \geq 1$ and large w) can be most easily understood by studying a simple model pseudopotential $U_x(\mathcal{R})$ defined as $U_x(0) = \infty$, $U_x(1) = 1$, $U_x(2) = x$, and $U_x(\mathcal{R}) = 0$ for $\mathcal{R} > 2$. This choice of U_x guarantees that skyrmions are its only finite-energy eigenstates, and their energy spectrum $E_S(K)$ depends on one free parameter x .

The essential information about the skyrmion wave functions is contained in the fractional grandparentage coefficients^{28,29} \mathcal{G} . The function $\mathcal{G}(\mathcal{R})$ is a pair-correlation function that gives the fraction of the total number of $e-e$

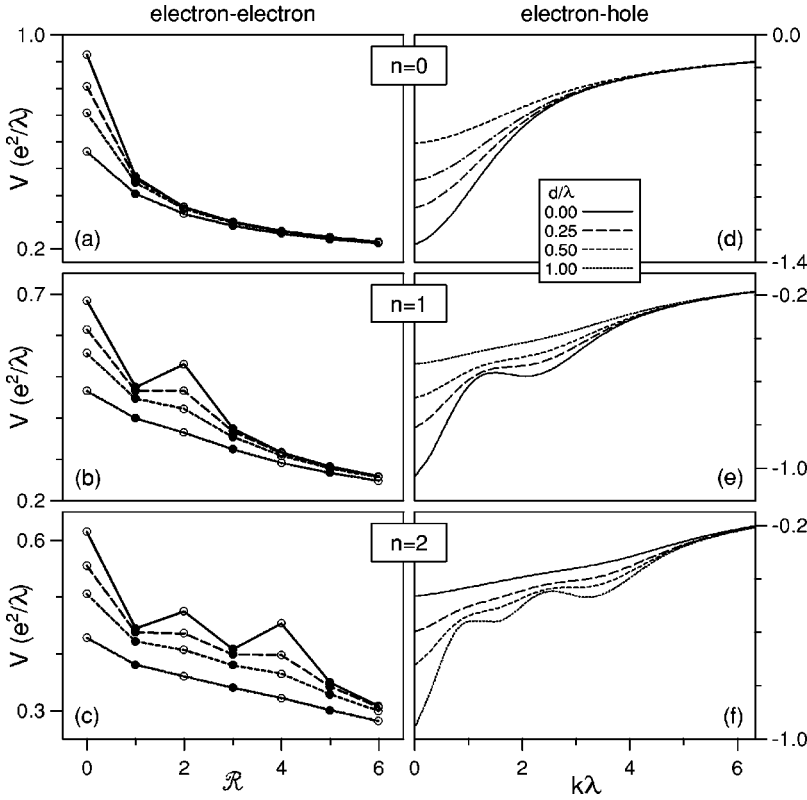


FIG. 14. Pseudopotentials V of the e - e (a)–(c) and e - h (d)–(f) interaction in the $n=0$ [(a),(d)], 1 [(b),(e)], and 2 [(c),(f)] LL's, calculated using the interaction potential $V_d(r) = e^2 / \sqrt{r^2 + d^2}$ for $d/\lambda = 0, \frac{1}{4}, \frac{1}{2},$ and 1, and describing a quasi-2D layer of width $w = 5d$. \mathcal{R} is relative e - e angular momentum (data shown only for $\mathcal{R} \leq 6$), open and closed circles mark singlet and triplet e - e states, respectively, k is the e - h wave vector, and λ is the magnetic length.

pairs with the relative pair angular momentum \mathcal{R} . For $K=0$, the many-electron system is completely spin-polarized, so that every electron pair has spin $S=1$, and thus \mathcal{G} vanishes for all even values of \mathcal{R} . When K is increased, so does the total fraction of $S=0$ pairs $\mathcal{G}_K(0) + \mathcal{G}_K(2) + \dots$, which happens at the cost of a decreasing number of $S=1$ pairs, $\mathcal{G}_K(1) + \mathcal{G}_K(3) + \dots$. The grandparentage coefficients measured from the “reference” value \mathcal{G}_0 corresponding to $K=0$ are plotted in Fig. 15(a) as a function of \mathcal{R} for $K=1, 2,$ and $\frac{1}{2}N$. It turns out that $\Delta\mathcal{G}_K = \mathcal{G}_K - \mathcal{G}_0$ is a regular function of \mathcal{R} and, for example, $\Delta\mathcal{G}_K(1) = -\alpha\mathcal{G}_K(2)$, where $\alpha^{-1} = 2 - (N-1)^{-1}$. This allows using a general expression for the interaction energy^{28,29}

$$E = \frac{1}{2}N(N-1) \sum_{\mathcal{R}} \mathcal{G}(\mathcal{R})V(\mathcal{R}) \quad (3)$$

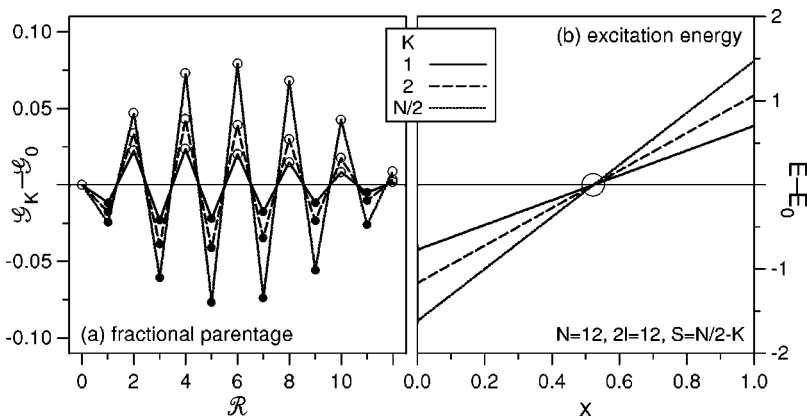


FIG. 15. (a) Pair-correlation functions—fractional grandparentage \mathcal{G}_K as a function of relative pair angular momentum \mathcal{R} —for skyrmions with $K=1$ and $\frac{1}{2}N$, calculated for $N=12$ electrons on Haldane sphere. Open and closed circles mark singlet and triplet pair states, respectively. (b) Energy E of the skyrmions with $K=1, 2,$ and $\frac{1}{2}N$ as a function of parameter x of the model e - e interaction U_x .

to write the skyrmion energy for $V=U_x$ as

$$E_S(K) - E_0 = (x - \alpha)\mathcal{G}_K(2). \quad (4)$$

As shown in Fig. 15(b), $E_S(K) - E_0$ changes sign simultaneously for all K (the spin-polarized QP state at $K=0$ becomes unstable toward formation of a skyrmion) when $x = \alpha$, that is when $U_x(2)$ drops below $\alpha U_x(1)$.

Since $\alpha \rightarrow \frac{1}{2}$ for $N \rightarrow \infty$, this means that skyrmions in an infinite (planar) system interacting through an arbitrary pseudopotential $V(\mathcal{R})$ will have lower energy than a QP state when V becomes superlinear between $\mathcal{R}=1$ and 3, that is when $V(1) - V(2) > V(2) - V(3)$. Owing to the linear (in an infinite system) relation^{6,28,29} between \mathcal{R} and the average squared distance $\langle r^2 \rangle$, this criterion can be rephrased as that V must be *superharmonic* (i.e., it must decrease more quickly than linearly as a function of $\langle r^2 \rangle$ between $\mathcal{R}=1$ and 3).

VIII. CONCLUSION

We have presented the results of detailed numerical studies of the spin excitations of various ferromagnetic GS's of a 2DEG confined in a quantum well of finite width w , in the integral and fractional quantum Hall regime. The calculations consist in the exact diagonalization of the Hamiltonian matrix of up to $N=14$ interacting electrons in Haldane spherical geometry, neglecting excitations to higher orbital LL's or to higher QW subbands.

Similar low-energy spectra in the vicinity of $\nu=1$ and $\frac{1}{3}$ ($\nu^*=1$ in the CF picture) have been found, and both contain the following two types of low-energy excitations: spin waves,⁷³ skyrmions, and antiskyrmions.^{19,59} The phase diagrams for the occurrence of skyrmions with different numbers of reversed spins K as a function of the well width w and the Zeeman energy E_Z have been determined at $\nu=1, 3$, and 5.

The interactions between the (neutral) spin waves and (charged) skyrmions have also been studied by exact diagonalization of the Hamiltonians describing one or more of these objects present in the underlying incompressible quantum Hall system. We propose that the spin waves each carrying angular momentum $L=1$ condense into an ordered (with parallel angular momenta), correlated, and noninteracting state. The interaction energy E_W of this condensate is a linear function of (continuous) spin polarization ζ which, in the absence of the Zeeman energy, gives rise to a gapless and continuous density of states. This is in contrast to a discrete spectrum of particlelike skyrmion excitations, whose energy E_S is a function of the (integral) reversed-spin number K . The short-range repulsion between charged skyrmions is predicted to cause Laughlin correlations, that is, the tendency to avoid skyrmion pair eigenstates with the smallest relative angular momenta \mathcal{R} . This causes the effective spatial isolation of skyrmions from one another and the absence of high-energy skyrmion-skyrmion collisions in the "dilute" regime at $|1-\nu|<\delta$ or $|1-\nu^*|<\delta^*$ (even in the fluid phase, beyond the melting point of the Wigner crystal) and the dependence of the average skyrmion size $\langle K \rangle$ on the filling factor in the opposite, incompressible regime.

The major differences between the $\nu=1$ and $\frac{1}{3}$ spectra are the reduced energy scale and a broken skyrmion-antiskyrmion symmetry in the latter system (broken particle-hole symmetry in the lowest CF LL). A number of phenom-

ena associated with the particular form of spin excitations at $\nu \approx 1$ (rapid depolarization at $\nu=1^\pm$, nonlinear transport through a finite-size droplet, sensitivity of the spin coupling to magnetic particles to ν , etc.) are also expected at $\nu \approx \frac{1}{3}$. The smallest skyrmion and antiskyrmion states at $\nu=\frac{1}{3}$ are equivalent to composite fermion charged excitons X_{CF}^\pm .

A qualitatively different behavior is observed in higher LL's. In agreement with earlier theories,^{60,61} we find that skyrmions and antiskyrmions are unstable at $\nu=3$ or 5 even at $E_Z=0$, which (in contrast to $\nu=1$ or $\frac{1}{3}$) results in the stability of the ferromagnetic order at all nearby values of ν , and the single-particle character of elementary excitations. In the e - h picture, this means unbinding of charged excitons in higher LL's. Also in contrast to $\nu=1$ or $\frac{1}{3}$, the $L=1$ spin waves attract one another rather than decouple, which for example results in direct confinement-induced transitions between the consecutive "compact" states of finite $\nu=3$ or 5 quantum Hall droplets, skipping the correlated, depolarized states with intermediate density.

This different behavior in higher LL's is suppressed when the width w of a quasi-2D layer exceeds about two magnetic lengths. This critical value obtained from a finite-size calculation seems to agree better with the experiments^{62,64} than an earlier estimate.⁶³ The reoccurrence of skyrmions in higher LL's in wider quantum wells is explained by studying the involved particle-particle and particle-hole interaction pseudopotentials and the electron correlations in the skyrmion eigenstates.

A criterion is found that allows the prediction of the presence or absence of skyrmions in a system of interacting spin- $\frac{1}{2}$ fermions in a half-filled spin-degenerate shell. The criterion describes correctly all calculated spin excitation spectra at $\nu=\frac{1}{3}, 1, 3$, and 5, and at arbitrary layer widths w , density profile $\varrho(z)$, etc.

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