# Skyrmions in the Moore-Read State at $\nu = \frac{5}{2}$

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(Received 21 October 2009; published 23 February 2010)

We study charged excitations of the non-Abelian Moore-Read liquid at a filling factor $\nu = \frac{5}{2}$, allowing for spin depolarization. Using a combination of numerical studies, and taking account of nonzero well widths, we find that at a sufficiently low Zeeman energy it is energetically favorable for charge $e/4$ quasiholes to bind into Skyrmions of charge $e/2$. We show that Skyrmion formation is further promoted by disorder, and argue that this can lead to a depolarized $\nu = \frac{5}{2}$ ground state in realistic experimental situations. We comment on the consequences for the activated transport.

DOI: 10.1103/PhysRevLett.104.086801 PACS numbers: 73.43.−f, 71.10.Pm

The $\nu = \frac{5}{2}$ quantum Hall state [1,2] has been intensively studied in recent years, mostly due to accumulating evidence that it realizes a non-Abelian phase of matter—the Moore-Read state [3]—which could serve as a platform for topological quantum computation [4]. A crucial step toward identifying the experimental $\nu = \frac{5}{2}$ state as the Moore-Read phase [3,5,6] is to demonstrate that the electron spins are fully polarized. Numerical simulations of model systems point strongly to such a polarized ground state [7–9]. However, the experimental situation remains unclear [2]. This issue has been reopened by recent experiments showing evidence of a depolarized ground state [10] and emphasizing [11] the unusual parallel field dependence of the activation energy [2]. Resolving this discrepancy has suddenly emerged as an important challenge.

In this Letter, we study spin polarization and charged excitations of the half-filled second Landau level. An important factor known to influence spin dynamics in this level [12] but remaining unexplored in the context of $\nu = \frac{5}{2}$ is the nonzero width $w$ of the quasi-two-dimensional electron layer. Indeed, the Coulomb repulsion for a spin-singlet pair of electrons softens considerably (relative to spin-triplet) upon increasing $w$ to the order of a magnetic length $\lambda \equiv \sqrt{\hbar c/eB}$. The results of our exact diagonalization (ED) studies [13] show that while the $\nu = \frac{5}{2}$ state remains spin polarized, its charged excitations can become depolarized Skyrmions at the experimentally relevant widths $w \gtrsim \lambda$.

Skyrmions at $\nu = \frac{5}{2}$ are qualitatively different from those studied previously at $\nu = 1$ or $\frac{1}{3}$ [14,15]. Their charge is twice that of the fundamental excitation, $q = e/4$. Thus, the $2q$-charged Skyrmion at $\nu = \frac{5}{2}$ can be viewed as a bound pair of like quasiparticles (QPs), held together against their Coulomb repulsion by a spin texture. This binding of QPs into Skyrmions has the striking consequence that Skyrmion formation is *promoted* by disorder: disorder can act to trap and confine QPs to a potential well, thereby facilitating their binding into Skyrmions. We show that this process can lead to a depolarized ground state under realistic experimental conditions. We suggest that this may account for the experimental observations of a reduction of polarization [10], and may play a role in the parallel field dependence of the activation energy [2,11].

Skyrmions in (ferromagnetic) quantum Hall liquids are topological excitations in which the local spin orientation wraps once over the surface of the spin sphere. The coupling of spin and orbital degrees of freedom causes a Skyrmion to appear as a single effective magnetic flux quantum, leading to a net charge $\nu' e$ where $\nu'$ is the filling fraction of the partially occupied Landau level [14]. (To be definite, we use the term "Skyrmion" to denote the excitation with a net reduction of particle number, and "anti-Skyrmion" for the excitation with a net increase.) At $\nu = \frac{5}{2}$ the partially occupied second Landau level ($LL_1$) is half-filled, so the Skyrmion has charge $e/2$. For vanishing Zeeman energy, the energy of a large Skyrmion is $\varepsilon_{\rm sky} = 4\pi\rho_s$, and the spin stiffness $\rho_s$ can be found from the dispersion of a single spin wave. We used ED on a sphere to calculate the energy spectra of $N$ electrons in $LL_1$, with one spin flip, at the magnetic flux $N_\phi = 2N - 3$ corresponding to the Moore-Read polarized ground state. Long-wavelength spin waves have quadratic dispersion, $E_{N/2-1}(L) = E_{N/2}(0) + 8\pi N^{-1}\rho_s L(L+1)$, where $E_S(L)$ is the $N$-electron energy as a function of total spin $S$ and angular momentum $L$. As (for the numerically accessible $N \le 16$) only the $L = 1$ spin wave state falls below the continuum of polarized excitations, we estimate $\rho_s$ by regression of $E_{N/2-1}(1)$ vs $N_\phi^{-1}$, as shown in Fig. 1(a).

It is important to note that the Skyrmion, or anti-Skyrmion, has twice the charge of the elementary quasihole (QH), or quasielectron (QE). Therefore, in order to gauge stability, the Skyrmion (anti-Skyrmion) energy $\varepsilon_{\rm sky}$ must be compared to the energy of a QH (QE) pair created in the same polarized ground state by adding (removing) one flux quantum, but without causing depolarization. The relevant quantity in this context is the "neutral" QP energy

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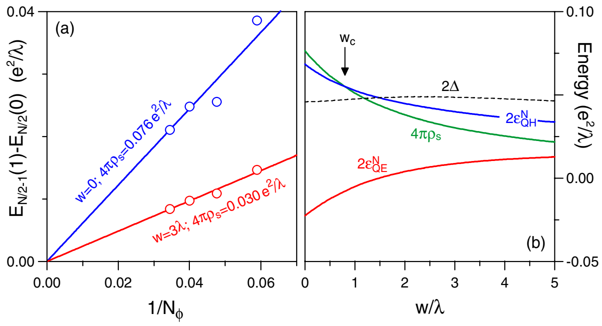


FIG. 1 (color online). (a) Extraction of the spin stiffness $\rho_s$ at $\nu = \frac{5}{2}$ from the long-wavelength part of the spectra $E_S(L)$ with a single spin flip ($E$, $S$, and $L$ are the energy, spin, and angular momentum of $N \leq 16$ electrons on a sphere, at flux $N_\phi = 2N - 3$). (b) Comparison of the Skyrmion energy $4\pi\rho_s$ and the neutral QP energies $\varepsilon^N$, establishing the Skyrmion as the lowest positively charged excitation for sufficient layer widths $w$ ($\lambda$ is the magnetic length; $\Delta = \varepsilon_{QH} + \varepsilon_{QE}$).

$\varepsilon^N$ [16–18]. Using the ground state and "proper" QP energies at $\nu = \frac{5}{2}$ [19,20] we find the width dependencies of $\varepsilon^N_{QH}$ and $\varepsilon^N_{QE}$ (with similar relations to the gap $\Delta = \varepsilon_{QH} + \varepsilon_{QE}$ as reported earlier for other states in $LL_1$ [17]). The comparison of $\varepsilon_{sky}(w)$ and $2\varepsilon^N_{QP}(w)$ in Fig. 1(b) yields the prediction that a critical width $w_c$ of the order of $\lambda$ is sufficient to induce transition from QHs to Skyrmions as the lowest energy positive excitations at $\nu = \frac{5}{2}$. In contrast, for all widths $w$ we have studied, our results show that the anti-Skyrmion has higher energy than a QE pair.

Since QHs of the Moore-Read state exhibit non-Abelian statistics, the binding of two QHs into a Skyrmion allows for two distinct fusion channels (1 and $\psi$) [21]. The case described above, in which the Skyrmion has energy $4\pi\rho_s$, refers to the vacuum fusion channel "1" in which the spin texture consists of the Moore-Read ground state with no unpaired fermions. Pairing in the $\psi$ channel corresponds to a Skyrmion spin texture in the presence of an additional neutral fermion excitation (in a finite system, this arises for an odd number of particles). As we have confirmed by ED for $N \leq 20$, for Skyrmion configurations (with two bound QHs) this unpaired neutral fermion has a positive energy. Thus, the fusion of QHs into a Skyrmion has lower energy in the 1 channel. In view of this, we shall focus mainly on this fusion channel, using finite-size studies with even $N$.

The above considerations apply in the case of vanishing Zeeman energy $E_Z$, when the Skyrmion has infinite size. In order to understand the properties at $E_Z > 0$, we have conducted extensive ED studies of depolarized states at $\nu = \frac{5}{2}$, using the spherical geometry [22], with Coulomb repulsion modified by the layer width $w$ [13]. An (infinite) Skyrmion is represented on a sphere by a maximal spin texture which uniformly covers its surface. In order to compute the ground states with the proper quantum numbers $L = S = 0$ we used a projected Lanczos procedure, carried out in the subspace defined by $(L_z, S_z)$ but restricted to the relevant eigensubspace $(L, S)$. This enabled efficient generation of the spin eigenstates with dimensions reaching $1.4 \times 10^9$ for $N = 12$ and $N_\phi = 26$.

Skyrmions (anti-Skyrmions) can be anticipated at the values of flux $N_\phi$ one quantum above (below) a polarized "parent" ground state. For $\nu = \frac{5}{2}$ this parent could belong to the universality class of either the Moore-Read (Pfaffian) state $\Phi_{MR}$ or its particle-conjugate ("anti-Pfaffian") $\Phi^*_{MR}$, on a sphere characterized by the "shifts" $\sigma \equiv 2N - N_\phi = 3$ and $-1$, respectively. This gives four possible Skyrmion candidates $\Psi_\sigma$ at $\sigma = 4, 2, 0, -2$ [23]. Indeed, in the energy spectra $E_S(L)$ obtained for $N \leq 12$ (and for different widths $w \leq 3\lambda$) we find that only at those four shifts are the lowest $S = 0$ states consistently nondegenerate ($L = 0$). Furthermore, at each width $w$, their ground state energies per particle all extrapolate to the same value as that of their polarized parents.

Features of the correlation functions of $\Psi_\sigma$ fully support their interpretation as Skyrmions. As shown on two examples in Fig. 2(a), their charge pair correlation functions $g = g_{\uparrow\uparrow} + g_{\uparrow\downarrow}$ are virtually identical to those of their polarized parents. Moreover, an extended spin texture is clearly revealed in a smoothly decreasing spin correlation function $f = g_{\uparrow\uparrow}/g$ (an initial increase of $f$ at short range being irrelevant due to the vanishing $g$). Our results for $\sigma = 2$ are consistent with those of Morf [7].

Focusing now on $\Psi_2$, let us compare it to the following trial wave function of an ideal Skyrmion. At $\nu = 1$, the Skyrmion wave function $\Psi_{sky}$ can be written as the wave function $\Phi$ of a polarized filled LL times an additional factor describing the spin texture [24]. The latter factor formally represents a many-body state of spinful bosons experiencing the (single quantum of) flux added to the $\nu = 1$ parent, and it is uniquely defined by $L$ and $S$. Here, we extend this idea to a polarized parent at $\nu = \frac{5}{2}$, in the following chosen to be the Moore-Read state $\Phi_{MR}$ [25]. The Skyrmion state $\Psi_{sky}$ is constructed from $\Phi$ by attaching a unique spin texture. The expression in the spherical coordinates $u_i$ and $v_i$ [22] is fairly simple

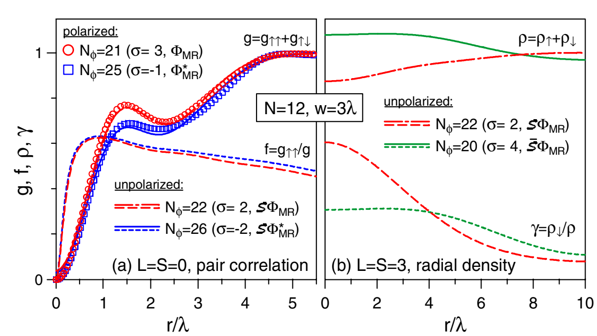


FIG. 2 (color online). (a) Charge and spin pair correlation functions ($g$ and $f$) of Skyrmion states $\Psi_2 = \mathcal{S}\Phi_{MR}$ and $\Psi_{-2} = \mathcal{S}\Phi^*_{MR}$ [23] at $\nu = \frac{5}{2}$, compared to their Pfaffian parents, calculated for $N = 12$ electrons and layer width $w = 3\lambda$. (b) Single-electron charge and spin-flip densities ($\varrho$ and $\gamma$) of the finite-size Skyrmions with an intermediate size $K = N/4$.

$$\Psi_{\rm sky}(\{u_i, v_i\}) = \mathcal{P}\left[\Phi(\{u_i, v_i\}) \times \begin{pmatrix} u_i \\ v_i \end{pmatrix}\right]. \quad (1)$$

Before projection $\mathcal{P}$ onto the global singlet $S = 0$, this state describes a radial spin texture and (like at $\nu = 1$) combines eigenstates with different $L = S$. The squared overlaps of $\Psi_{\rm sky}$ with the unpolarized Coulomb ground state $\Psi_2$ were calculated in a standard way [26]. The values for $\Psi_{\rm sky}$ derived from $\Phi_{\rm MR}$ are given in Table I.

We have also constructed an alternative trial Skyrmion state $\Psi'_{\rm sky}$, as the ground state of a model Hamiltonian $H_{\rm sky} = V_0(0) + W_{3/2}(3) + \epsilon V_0(2)$, where $\epsilon \ll 1$, and $V_S(m)$ and $W_S(m)$ are the pair and triplet pseudopotentials. The resulting state, $\Psi'_{\rm sky}$, is essentially the same as $\Psi_{\rm sky}$ defined above (squared overlap of 0.96 for $N = 12$). By tracing the successive action of each term we found that the (moderate) overlaps with $\Psi_2$ are largely lost at the stage of enforcing triplet correlations. This reflects the fact that the Pfaffian $\Phi_{\rm MR}$ is not a very accurate description of the Coulomb ground state $\Phi_3$ [6,8], here acting as the Skyrmion's parent. Indeed a comparison (Table I) of the overlaps of $\Psi'_{\rm sky}$ or $\Psi_{\rm sky}$ with the exact Coulomb Skyrmion state shows that the reduction in total overlap is comparable to that between $\Phi_{\rm MR}$ and $\Phi_3$.

We have also examined the partially polarized spectra at $\sigma = 4, 2, 0, -2$. Finite Skyrmions of size $K \leq N/2$ on the sphere have $L = S = N/2 - K$ [24]. As shown for a few examples in Fig. 2(b), the plots of single-particle charge and spin-flip densities ($\varrho = \varrho_\uparrow + \varrho_\downarrow$ and $\gamma = \varrho_\downarrow/\varrho$) calculated in the lowest Coulomb states at $L_z = L = S_z = S = 1, 2, \ldots$ indeed reveal accumulation of charge $2q$ and spin $K$ over a finite area $\propto K$ around a pole.

We now turn to study the competition between the finite Skyrmions and the spinless QPs. The analogy with $\nu = 1$ or $\frac{1}{3}$ fails, as Skyrmions at $\nu = \frac{5}{2}$ carry two charge quanta ($2q = e/2$), which allows their spontaneous breakup into pairs of repelling QPs. In order to resolve the issue of stability, we have carefully compared the Skyrmion and 2QP energies, including the electrostatic corrections [19] to compensate for finite-size effects due to different spatial extent of the involved carriers [27].

As illustrated in Fig. 3(a) for $N = 12$ and $w = 3\lambda$, the Skyrmion–anti-Skyrmion asymmetry is very strong at $\nu = \frac{5}{2}$. In agreement with Fig. 1, large Skyrmions are energetically favored over the QHs, but large anti-Skyrmions have higher energy than the QEs. As expected, the finite Skyrmions characterized by $L = S$ are no longer the lowest states when spin polarization becomes too high. The emergence of lower-energy states at $K \lesssim N/4$ signals the breakup of a Skyrmion into two separate objects (whose counteraligned angular momenta give rise to the observed oscillation between $L = 0$ and 1). At full polarization, the two objects are Moore-Read QHs; below that, they are $q$-charged spin textures (CSTs) formed around the individual QHs [28,29].

For a clean (disorder-free) sample, we find that, while the $\nu = \frac{5}{2}$ ground state remains polarized at $E_Z = 0$, the nature of its charged excitations (in sufficiently wide wells) depends on $E_Z$. For large $E_Z$ the activation gap is set by the energy of a single (polarized) QE-QH pair. For small $E_Z$, these are replaced by CSTs. The minimal activation gap is never expected to involve a Skyrmion, as this requires creation also of two $-e/4$ QEs (or CSTs).

Skyrmions can become relevant in a clean sample when dilute QHs are introduced into the ground state by tuning to $\nu < 5/2$. From Fig. 3(b), dilute QHs will convert to CSTs for $E_Z \lesssim 0.01e^2/\lambda$ and then pair up into finite-sized Skyrmions for $E_Z \lesssim 0.003e^2/\lambda$. At $\nu > 5/2$, dilute QEs may convert into CSTs at small $E_Z$; however, they are not expected to bind into anti-Skyrmions. Since the Skyrmion is formed by binding two QHs, its stability may be *enhanced* by a nonzero QH concentration. Thus, with increasing the QH density at fixed $E_Z$, a Wigner crystal of QHs could convert into a Wigner crystal of Skyrmions.

Real experimental samples always involve some disorder. This can lead to Skyrmion formation in the ground state. Even at the center of the $\nu = \frac{5}{2}$ plateau, long-range disorder can nucleate puddles of QHs and QEs, which can evolve into CSTs or Skyrmions at small $E_Z$. In states such as $\nu = 1$ or $\frac{1}{3}$, disorder acts to disfavor Skyrmions, as it reduces the size of trapped carriers. Surprisingly, the effect of disorder at $\nu = \frac{5}{2}$ is the opposite: it can act to bring two repelling QHs together and therefore promote their merging

TABLE I. Squared overlaps of the trial Skyrmion states $\Psi_{\rm sky}$ and $\Psi'_{\rm sky}$ and their Pfaffian parents $\Phi_{\rm MR}$ with the exact Coulomb ground states (unpolarized $\Psi_2$ and polarized $\Phi_3$) for $N = 10$ electrons and layer widths $w = 0$ and $3\lambda$.

| $w/\lambda$ | $\lvert\langle\Psi_{\rm sky}\lvert\Psi_2\rangle\rvert^2$ | $\lvert\langle\Psi'_{\rm sky}\lvert\Psi_2\rangle\rvert^2$ | $\lvert\langle\Phi_{\rm MR}\lvert\Phi_3\rangle\rvert^2$ |
|---|---|---|---|
| 0 | 0.51(3) | 0.5186 | 0.7016 |
| 3 | 0.71(1) | 0.7394 | 0.8310 |

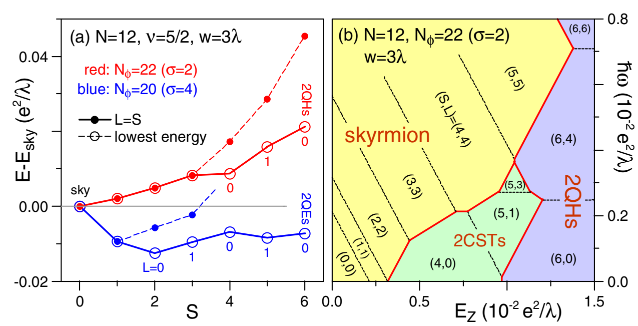


FIG. 3 (color online). (a) Dependence of energy $E$ (counted from $E_{\rm sky}$ at $S = 0$) on spin $S$, for $N = 12$ electrons at flux $N_\phi = 20$ and 22. Solid and dashed lines connect the ground states and the lowest $L = S$ states at each $S$. The energies include electrostatic corrections defined in the text. (b) Phase diagram in the plane of Zeeman energy $E_Z$ and lateral harmonic confinement $\hbar\omega$, showing transitions between various states: two $e/4$-charged quasiholes (QHs); two charged spin-textures (CSTs); or one Skyrmion ($L = S$).

into a Skyrmion. In order to investigate these effects, we have studied two QHs in a lateral harmonic confinement of strength $\hbar\omega$, which models the potential in a local minimum of the disorder. From the ED results for the ground state energy $E_S(L)$, we construct the phase diagram shown in Fig. 3(b). This shows that, as disorder strength ($\hbar\omega$) is increased, the Skyrmion is stable up to larger values of $E_Z$.

Our results show that, for GaAs samples where $\nu = \frac{5}{2}$ occurs at $B \lesssim 6$ T, QHs may bind into Skyrmions if assisted by disorder. This follows from Fig. 3(b), where the reduction of spin from its maximal value occurs for $E_Z \lesssim 0.014e^2/\lambda$. However, we emphasize that this numerical value is subject to significant uncertainty owing to finite-size effects. Still, our results make the clear qualitative point that, even at the center of the $\nu = \frac{5}{2}$ plateau, in samples with sufficiently low Zeeman energy, the ground state will not be fully polarized: QHs trapped by disorder will bind into Skyrmions causing depolarization. We suggest this could account for the experimental evidence of the lack of spin polarization at $\nu = \frac{5}{2}$ [10].

The appearance of this unpolarized ground state may have consequences for the activated transport. The situation is very different from $\nu = 1$ or $\frac{1}{3}$, where Skyrmions are the charged carriers and increasing $E_Z$ increases the activation gap. At $\nu = \frac{5}{2}$, Skyrmions can form in a (disordered) *ground state*. Activated transport will still involve the motion of $e/4$ excitations (the minimal charged object), but now hopping of QHs will have an activation energy that includes a contribution to overcome their binding into Skyrmions, which *decreases* with increasing $E_Z$. This effect could potentially account for the observed reduction of activation energy with *small* in-plane fields [2,11]. We further note that the tendency toward binding of topologically nontrivial QHs into topologically trivial Skyrmions could strongly affect interference experiments designed to probe non-Abelian statistics [30].

In conclusion, we have shown that for realistic widths $w \gtrsim \lambda$ the QH excitations of the $\nu = \frac{5}{2}$ Moore-Read state can bind to form Skyrmions at small Zeeman energy. Long-range disorder acts to bring trapped QHs together and to promote Skyrmion formation. This can lead to a depolarized ground state even at the center of the plateau. This may account for recent measurements of spin depolarization [10], and for the unusual in-plane field dependence of the activation energy [2,11].

We acknowledge the support of the EU under the Marie Curie Grant No. PIEF-GA-2008-221701 and the Polish MNiSW under Grant No. N202-071-32/1513 (A. W.); Trinity Hall (G. M.); and EPSRC under EP/F032773/1 (N. R. C.).

[1] R. Willett, J. P. Eisenstein, H. L. Stormer, D. C. Tsui, A. C. Gossard, and J. H. English, Phys. Rev. Lett. **59**, 1776 (1987).
[2] J. P. Eisenstein, R. Willett, H. L. Stormer, D. C. Tsui, A. C. Gossard, and J. H. English, Phys. Rev. Lett. **61**, 997 (1988).
[3] G. Moore and N. Read, Nucl. Phys. **B360**, 362 (1991).
[4] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. **80**, 1083 (2008).
[5] N. Read and D. Green, Phys. Rev. B **61**, 10 267 (2000).
[6] G. Möller and S. H. Simon, Phys. Rev. B **77**, 075319 (2008).
[7] R. H. Morf, Phys. Rev. Lett. **80**, 1505 (1998).
[8] E. H. Rezayi and F. D. M. Haldane, Phys. Rev. Lett. **84**, 4685 (2000).
[9] A. E. Feiguin, E. H. Rezayi, K. Yang, C. Nayak, and S. Das Sarma, Phys. Rev. B **79**, 115322 (2009).
[10] A. Pinczuk (private communication).
[11] C. R. Dean, B. A. Piot, P. Hayden, S. Das Sarma, G. Gervais, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett. **100**, 146803 (2008); **101**, 186806 (2008).
[12] N. R. Cooper, Phys. Rev. B **55**, R1934 (1997).
[13] We assume hard wall confinement in the perpendicular direction, yielding single-particle density $\propto \cos^2(\pi z/w)$.
[14] S. L. Sondhi, A. Karlhede, S. A. Kivelson, and E. H. Rezayi, Phys. Rev. B **47**, 16 419 (1993).
[15] A. H. MacDonald and J. J. Palacios, Phys. Rev. B **58**, R10 171 (1998).
[16] R. Morf and B. I. Halperin, Phys. Rev. B **33**, 2221 (1986).
[17] A. H. MacDonald and S. M. Girvin, Phys. Rev. B **34**, 5639 (1986).
[18] The "neutral" QP energy $\varepsilon^N$ differs from the more common "proper" QP energy $\tilde{\varepsilon}$ by an appropriate fraction (here, $\pm\frac{1}{8}$) of the ground state energy per particle.
[19] R. H. Morf, N. d'Ambrumenil, and S. Das Sarma, Phys. Rev. B **66**, 075408 (2002).
[20] A. Wójs, Phys. Rev. B **80**, 041104(R) (2009).
[21] M. Baraban, G. Zikos, N. Bonesteel, and S. H. Simon, Phys. Rev. Lett. **103**, 076801 (2009).
[22] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).
[23] The Skyrmions ($\mathcal{S}$) and anti-Skyrmions ($\bar{\mathcal{S}}$) in the Pfaffian ($\Phi_{\rm MR}$) and anti-Pfaffian ($\Phi^*_{\rm MR}$) parents are identified as $\Psi_4 = \bar{\mathcal{S}}\Phi_{\rm MR}$, $\Psi_2 = \mathcal{S}\Phi_{\rm MR}$, $\Psi_0 = \bar{\mathcal{S}}\Phi^*_{\rm MR}$, $\Psi_{-2} = \mathcal{S}\Phi^*_{\rm MR}$.
[24] A. H. MacDonald, H. A. Fertig, and L. Brey, Phys. Rev. Lett. **76**, 2153 (1996).
[25] The same construction could be generalized to other polarized states, e.g., an optimized weakly paired state [6], or the numerical Coulomb ground state $\Phi_3$.
[26] G. Möller, S. H. Simon, and E. H. Rezayi, Phys. Rev. B **79**, 125106 (2009).
[27] There is no correction at $S = 0$, as the charge distribution is uniform; a concentrated charge is assumed at $S = N/2$, as standard; a cubic interpolation is applied in between.
[28] I. Dimov, B. I. Halperin, and C. Nayak, Phys. Rev. Lett. **100**, 126804 (2008).
[29] A comparison of the ED results for the unpolarized states with $\sigma = 2$, for even and odd $N$ ($S = 0$ and $\frac{1}{2}$, respectively), suggests that the QH binding at small $E_Z$ occurs in the "1" fusion channel also for small Skyrmions.
[30] R. L. Willett, L. N. Pfeiffer, and K. W. West, Proc. Natl. Acad. Sci. U.S.A. **106**, 8853 (2009); (unpublished).