Second generation of Moore-Read quasiholes in a composite-fermion liquid

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Two- and three-body correlations of incompressible quantum liquids are studied numerically. Pairing of composite fermions (CFs) in the 1/3-filled second CF Landau level is found at $\nu_e=4/11$. It is explained by reduced short-range repulsion due to ring-like single-particle charge distribution. Although Moore-Read state of CFs is unstable in the 1/2-filled second CF level, condensation of its quasiholes is a possible origin of incompressibility at $\nu_e=4/11$. Electron pairing occurs at $\nu_e=7/3$ and 13/3, but with different pair-pair correlations. Signatures of triplets are found at higher fillings.

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Strong magnetic field B applied to a two-dimensional electron gas (2DEG) rearranges its single-particle density of states to a series of discrete Landau levels (LL_n). When the cyclotron gap $\hbar\omega_c \propto B$ exceeds Coulomb energy $e^2/\lambda \propto \sqrt{B}$ ($\lambda=\sqrt{\hbar c/eB}$ being the magnetic length), the low-energy dynamics depends on interactions in one, partially filled LL. Despite reminiscence to atomic physics, macroscopic degeneracy and a distinct scattering matrix lead to very different, fascinating behavior.[1]

Fractional quantum Hall effect[2] reveals plethora of highly correlated electron phases at various LL filling factors $\nu_e=2\pi\varrho\lambda^2$ (ϱ being the 2D concentration). Among them are Laughlin[3] and Jain[4] incompressible liquids (IQLs) with fractionally charged quasiparticles (QPs) at $\nu_e=\frac{1}{3}$ or $\frac{2}{5}$, Wigner crystals[5] at $\nu_e \ll 1$, and stripes[6] in high LLs. In addition to transport,[2] they are probed by shot-noise [allowing detection of fractional charge of the QPs (Ref. 7)] and optics (with discontinuities in photoluminescence energy related to the QP interactions[8]).

A key concept in understanding IQLs is Jain's composite fermion (CF) picture.[4] The CFs are fictitious particles, electrons that captured part of the external magnetic field B in form of infinitesimal tubes carrying an even number $2p$ of flux quanta $\phi_0=hc/e$. The most prominent IQLs at $\nu_e=n(2ps\pm1)^{-1}$ are represented by the completely filled s lowest LLs of the CFs ($\mathrm{CF\text{-}LL}_n$ with $n<s$) in a residual magnetic field $B^*=B-2p\phi_0\varrho$.

Not all IQLs are so easily explained by the CF model, e.g., Haldane-Rezayi[9] and Moore-Read[10] paired liquids proposed for $\nu_e=\frac{5}{2}$. Because of nonabelian statistics of its quasiholes (QHs), especially the latter state has recently stirred renewed interest as a candidate for quantum computation in a solid-state environment.[11]

Another family of IQLs discovered by Pan *et al.*[12] at $\nu_e=\frac{4}{11}$, $\frac{3}{8}$, and $\frac{5}{13}$ corresponding to fractional CF fillings $\nu_{\mathrm{CF}}=\nu_e(1-2p\nu_e)^{-1}=\frac{4}{3}$, $\frac{3}{2}$, and $\frac{5}{3}$ (with $p=1$). Assuming spin polarization, all these states have a partially filled $\mathrm{CF\text{-}LL}_1$. Their incompressibility results from residual CF–CF interactions. Familiar values of ν_{CF} suggested similarity between partially filled electron and CF LLs.[13] For $\nu_e=\frac{4}{11}$ and $\frac{5}{13}$, it revived the "QP hierarchy,"[14] whose CF formulation consists of the CF→electron mapping followed by reapplication of the CF picture in $\mathrm{CF\text{-}LL}_1$,[15] leading to a "second generation" of CFs.[16] However, this idea ignored the requirement of a strong short-range repulsion.[17,18] Indeed, it was later excluded in exact diagonalization studies,[19] in which a different series of finite-size $\nu_e=\frac{4}{11}$ liquids with larger gaps was identified. On the other hand, Moore-Read liquid of paired CFs was tested[20] for $\nu_e=\frac{3}{8}$, but it was eventually ruled out in favor of the stripe order.[21,22]

In this paper, we study two- and three-body correlations in several IQLs whose origin of incompressibility remains controversial. The main result is for the $\nu_e=\frac{4}{11}$ liquid, corresponding to the one-third filling of $\mathrm{CF\text{-}LL}_1$. We do not show why it is incompressible (a fact known from both experiment[12] and exact diagonalization[19]), but we find evidence for CF pairing in this state. By connection with the Moore-Read state of CFs at half filling, we interpret the $\nu_e=\frac{4}{11}$ state as a condensate of QHs of the "second generation" Moore-Read state of the CFs. Its pair-pair or QH-QH correlations are not defined, but a Laughlin form[23] is excluded. The importance of identification of the role of Moore-Read QHs in the $\nu_e=\frac{4}{11}$ liquid lies in their non-Abelian statistics.[10]

In Figs. 1(a) and 1(b) charge-density distributions of electrons are compared with three different CF quasiparticles at $\nu_e=\frac{1}{3}$. Laughlin liquid is a filled spin-polarized $\mathrm{CF\text{-}LL}_0$, and

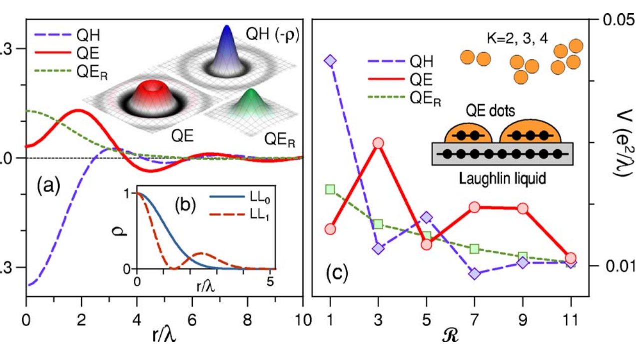

FIG. 1. (Color online) (a) Radial charge distribution profiles of different composite fermions: Laughlin quasielectron (QE), quasihole (QH), and reversed-spin quasielectron (QE_R); results obtained from exact 10-electron diagonalization; λ is the magnetic length. (b) Same for electrons in two lowest Landau levels. (c) Haldane pseudopotentials (interaction energy V vs relative pair angular momentum $\mathcal{R}$) for composite fermions; inset: schematic of "artificial composite fermion atoms."

its quasielectron (QE), quasihole (QH), and reversed-spin quasielectron (QE_R) correspond to a particle in CF-LL_1, a vacancy in CF-LL_0, and a spin-flip particle in CF-LL_0, respectively. Particles/holes in CF-LL_0 resemble those in LL_0. However, the ring structure in CF-LL_1 makes the QEs different from the electrons and causes strong reduction of the QE-QE repulsion at short range [see Fig. 1(c)]. Such interaction cannot[17,18] produce a Laughlin IQL of the QEs at the $\nu=\frac{1}{3}$ filling of CF-LL_1. Instead, other QE-QE correlations must be considered.

Spontaneous QE cluster formation would be somewhat analogous to the self-assembled growth of strained quantum dots.[24] A full CF-LL_0 representing the uniform-density Laughlin liquid plays the role of a "wetting layer." Over this background, in analogy to atoms grouping into dots to minimize the elastic energy, QEs moving within CF-LL_1 arrange themselves into pairs or larger QE clusters easily pinned down by disorder. While in electronic "artificial atoms" the self-organization of real atoms serves a purpose of external confinement for the electrons, in their CF analogs both these roles are played by the QEs. Another distinction is the fractional charge of bound QE carriers. A similar electron-atom analogy was earlier explored in the context of condensed states of cold atoms in rotating harmonic traps.[25]

In numerics we considered $N\leq 12$ particles ($N=12$ being divisible by $K=2$, 3, and 4) of charge q ($-e$ for electrons and $-\frac{1}{3}e$ for CFs) confined to a sphere[14] of radius R. For its high symmetry, this geometry is especially useful in studying quantum liquids, while the alternative choice of periodic boundary conditions (torus) is more appropriate for broken-symmetry phases. The radial magnetic field B is created by a Dirac monopole of strength $2Q=4\pi R^2 B\phi_0^{-1}$. The single-particle LLs are distinguished by shell angular momentum $l\geq Q$.

As for a partially filled atomic shell, the many-body Hamiltonian on a sphere is determined by particle number N, shell degeneracy $g=2l+1$, and interaction matrix elements. Using Clebsch-Gordan coefficients, the latter are related to Haldane[26] pseudopotentials V_L (energies of pairs with angular momentum L). The pseudopotential combines information about the potential $v(r)$ and shell wavefunctions, so it may not be similar in different systems with the same (Coulomb) forces. In macroscopic quantum Hall systems, only the ratio $\nu=N/g$ (filling factor) is important, and V is a function of relative pair angular momentum $\mathcal{R}=2l-L$ (for fermions, an odd integer). The strategy in exact diagonalization is therefore to study different finite systems $(N,2l)$ with a realistic interaction $V(\mathcal{R})$, in search of those properties which scale properly with size and persist in the macroscopic limit.

In the following we will distinguish $\nu_e=2\pi\varrho\lambda^2$ from the effective filling factor $\nu=N/g<1$ of only those electrons or CFs in their highest, partially filled shell. In LL_n, $\nu_e=2n+\nu$. In CF-LL_n (assuming spin-polarization) $\nu_{CF}=n+\nu$ and $\nu_e=\nu_{CF}(2p\nu_{CF}+1)^{-1}$.

The CF pseudopotentials shown in Fig. 1(c) were obtained using a similar method to Ref. 21, by combining short-range data from exact diagonalization[18] with long-range behavior of point charges $\pm\frac{1}{3}e$. Weak QE-QE repulsion at $\mathcal{R}=1$ is the reason why the $\nu=\frac{1}{3}$, $\frac{2}{3}$, and $\frac{1}{2}$ states of QEs are not the "second generation" Laughlin, Jain, or Moore-Read states (of QEs). The average QE-QE interaction energies (per particle) in these states overestimates the actual QE eigenenergies by at least $0.003e^2/\lambda$ (6–7 %). Clearly, the microscopic origin of the observed QE incompressibility must be different.

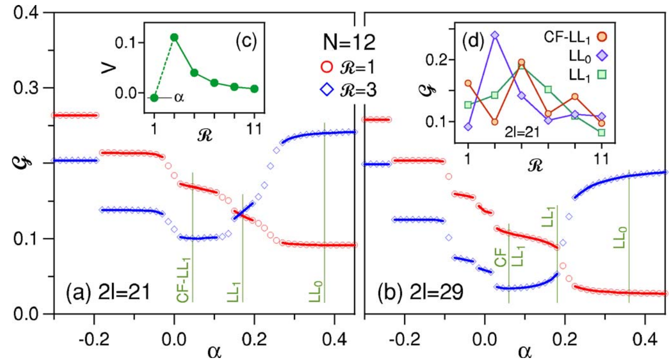

FIG. 2. (Color online) Haldane pair amplitudes $\mathcal{G}$ (~ number of pairs) at relative pair angular momenta $\mathcal{R}=1$ and 3, of $N=12$ fermions in angular momentum shells with $2l=21$ (a) and $2l=29$ (b), as a function of parameter α of the interaction pseudopotential shown in (c). (d) Amplitudes $\mathcal{G}(\mathcal{R})$ of electrons and composite fermions in different Landau levels.

What are these known correlations, excluded for QEs? Laughlin correlations result from strong short-range repulsion (such as between electrons in LL_0). They consist of the maximum avoidance of pair states with the smallest $\mathcal{R}$. For example, Laughlin $\nu=\frac{1}{3}$ state is the zero-energy ground state of a model pseudopotential $V=\delta_{\mathcal{R},1}$.[14] For more realistic interactions, the exact criterion is that V must rise faster than linearly when $\mathcal{R}$ decreases.[18] A linear decrease of V between $\mathcal{R}=1$ and 5 (such as in LL_1) leads to different correlations. For example, Moore-Read $\nu=\frac{1}{2}$ liquid involves pairing and Laughlin correlations among pairs. It is the zero-energy ground state of a model three-body pseudopotential $V=\delta_{\mathcal{T},3}$ ($\mathcal{T}=3l-L\geq 3$ is the relative triplet angular momentum, proportional to the area spanned by three particles).[10] This is an example of dynamics induced by real two-body forces, described more easily by an effective three-body interaction.

Weak QE-QE repulsion at $\mathcal{R}=1$ compared to $\mathcal{R}=3$ could force QEs into even larger clusters. As a simple classical analogy, consider a string of point particles, one per unit length, with a repulsive potential $v_a(r)=a+(1-a)r$ for $r<1$ and $1/r^2$ otherwise. Equal spacing is favored for $a>1.64$, and transitions to pairs, triplets, and larger clusters occur for decreasing a. A similar rearrangement might occur when going from LL_0 to LL_1 and CF-LL_1, with $V(1)$ playing the role of $v_a(0)\equiv a$.

In Fig. 2 we plot two leading "Haldane amplitudes"[26] $\mathcal{G}(1)$ and $\mathcal{G}(3)$. The discrete pair-correlation function $\mathcal{G}(\mathcal{R})$ is proportional to the number of pairs with a given $\mathcal{R}$ and normalized to $\Sigma_{\mathcal{R}}\mathcal{G}(\mathcal{R})=1$. It connects many-body interaction energy with a pseudopotential, $E=\binom{N}{2}\Sigma_{\mathcal{R}}\mathcal{G}(\mathcal{R})V(\mathcal{R})$. Here, $\mathcal{G}$ is calculated in the ground states of $N=12$ particles at $2l=21$ and 29 (corresponding to $\nu=\frac{1}{2}$ and $\frac{1}{3}$ for the QEs[19]) with model interaction shown in the inset: $V_\alpha(1)=\alpha$ and $V_\alpha(\mathcal{R}>1)=1/\mathcal{R}^2$. At $\alpha>0.3$, $\mathcal{G}(1)$ takes on the minimum possible

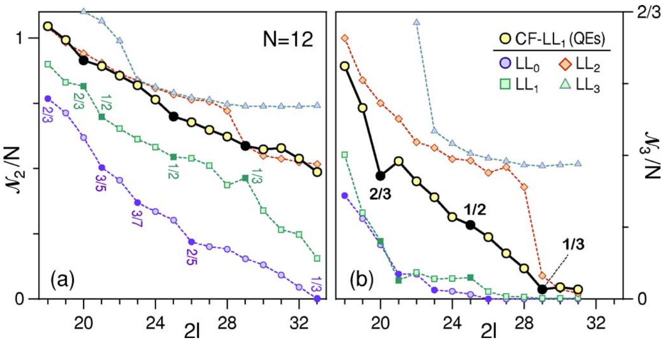

FIG. 3. (Color online) Number of pairs $\mathcal{N}_2$ (a) and triplets $\mathcal{N}_3$ (b) with the minimum relative angular momentum ($\mathcal{R}=1$ or $\mathcal{T}=3$) for $N=12$ electrons or composite fermions in angular momentum shells with $2l=18$ to 33, corresponding to fractional Landau level fillings $\frac{1}{3}\leq\nu\sim N/(2l+1)\leq\frac{2}{3}$. Finite-size incompressible states are labeled by ν.

value, which means Laughlin correlations (no clusters). At $\alpha<-0.25$, $\mathcal{G}(1)$ reaches maximum, and the particles form one big $\nu=1$ quantum Hall droplet (QHD). The transition between the two limits occurs quasidiscontinuously through a series of well-defined states seen as plateaus in $\mathcal{G}(\alpha)$.

The cluster size K cannot be assigned to each state because the number of plateaus depends on the choice of V_α. The comparison of $\mathcal{G}(1)$ with the values predicted for N/K independent QHDs of size $K=2$, 3, and 4 is not convincing because in a few-cluster system each QHD is relaxed by the cluster-cluster interaction, lowering $\mathcal{G}(1)$. Another problem is the contribution to $\mathcal{G}(1)$ from pairs of particles belonging to different clusters. Nevertheless, it is clear that the "degree of clustering" changes as a function of α in a quantized fashion, supporting the picture of N particles grouping into various clustered configurations. Furthermore, the values of α for which V_α reproduces the exact ground states of QEs or electrons belong to different continuity regions, confirming different correlations in LL_0, LL_1, and CF-LL_1 (except for a possible similarity of the $\nu=\frac{1}{3}$ states in LL_1 and CF-LL_1).

In Fig. 3(a) we compare $\mathcal{N}_2=\binom{N}{2}\mathcal{G}(1)$, the number of pairs with $\mathcal{R}=1$, calculated in the ground states of $N=12$ CFs and electrons as a function $2l$. The downward cusps in $\mathcal{N}_2(2l)$ at a series of Laughlin/Jain states in LL_0 are well understood. We also marked $2l=2N-3$ and $3N-7$ corresponding to incompressible $\nu=\frac{1}{2}$ and $\frac{1}{3}$ ground states in LL_1 and CF-LL_1,[19] and their particle-hole conjugates ($N\to g-N$) at $2l=2N+1$ and $\frac{3}{2}N+2$.

The comparison of $\mathcal{N}_2$ tells about short-range pair correlations in different LLs. There are significantly more pairs in CF-LL_1 and in excited electron LLs than in LL_0. In LL_1, the Moore-Read state is known to be paired, and indeed $\mathcal{N}_2\approx\frac{1}{2}N$ at $\nu=\frac{1}{2}$. A similar value is obtained for the (not well understood) $\nu=\frac{1}{3}$ state at $2l=29$. At $2l>29$ the number of pairs drops roughly linearly with $2l$, aiming at zero for $2l\sim 35$, suggesting Laughlin correlations in LL_1 at sufficiently low filling. The CF-LL_1 is different (in terms of $\mathcal{N}_2$) from LL_0 or LL_1 in the whole range of $18\leq 2l\leq 33$. However, it appears similar to LL_2 at both $2l\leq 23$ and $2l\geq 29$. Also, LL_2 and LL_3 look alike for $23\leq 2l<29$. While convincing assignment of ν to a finite state $(N,2l)$ requires studying size dependence (we looked at different $N\leq 12$), notice that $N/g=\frac{1}{2}$ at $2l=23$, and $2l=29$ is the $\nu=\frac{1}{3}$ state in LL_1 and CF-LL_1. On the other hand, comparison with $N\neq 12$ does not allow for the assignment of any particular ν to $N=12$ and $2l=35$. Note also that similar short-range correlations in different states do not guarantee high overlaps of their wave functions. Here, only $\langle LL_2|LL_3\rangle^2$ reaches 0.67 while all other overlaps, including $\langle QE|LL_n\rangle^2$, essentially vanish.

In Fig. 3(b) we plot $\mathcal{N}_3$, the number of "compact" triplets with $\mathcal{T}=3$. It is proportional to the first triplet Haldane amplitude and tells about short-range three-body correlations. In both LL_0 and LL_1, $\mathcal{N}_3$ decreases roughly linearly as a function of $2l$ and drops to essentially zero at $2l=21$, the smallest value at which the $\mathcal{T}=3$ triplets can be completely avoided. Exactly $\mathcal{N}_3=0$ would indicate the Moore-Read state, but its accuracy for the actual $\nu=\frac{1}{2}$ ground state in LL_1 depends sensitively on the quasi-2D layer width and on the surface curvature.[27,28] Nevertheless, clusters larger than pairs clearly do not form in neither LL_0 nor LL_1 at $\nu\leq\frac{1}{2}$.

The number of QE triplets in CF-LL_1 is also a nearly linear function of $2l$, but it drops to zero at $2l=3N-7=29$, earlier identified with $\nu=\frac{1}{3}$ in this shell (i.e., with $\nu_e=\frac{4}{11}$).[19] In connection with having $\mathcal{N}_2\approx\frac{1}{2}N$ pairs, vanishing of $\mathcal{N}_3$ is the evidence for QE pairing at $\nu_e=\frac{4}{11}$. (The same argument was earlier used[28] to numerically demonstrate pairing at the half-filled LL_1.)

Two types of elementary excitations which appear in the paired $\nu=\frac{1}{2}$ Moore-Read state when $2l$ is increased beyond $2N-3$ are the $\frac{1}{4}q$-charged QHs (of the Laughlin liquid of pairs[23]) and pair-breaking neutral-fermion excitations.[10,27,28] Being paired, the QE state at $2l=3N-7$ can only contain the QHs but no pair breakers. The interaction of Moore-Read QHs in CF-LL_1 is not known, but evidently (as seen in experiment[12] and in numerics[19]) it causes their condensation into an incompressible liquid at $\nu=\frac{1}{3}$.

The "second generation" (to distinguish from $\nu_e=\frac{5}{2}$) Moore-Read state of QEs would occur at $\nu=\frac{1}{2}$ in CF-LL_1 (i.e., at $\nu_{CF}=\frac{3}{2}$ or $\nu_e=\frac{3}{8}$). Its instability[21,22] does not necessarily preclude reentrance with additional QHs at a lower ν and, in particular, their condensation at $\nu=\frac{1}{3}$ (i.e., at $\nu_{CF}=\frac{4}{3}$ or $\nu_e=\frac{4}{11}$). A similar situation occurs with Jain $\nu=\frac{2}{7}$ state, obtained (in Haldane hierarchy) from Laughlin $\nu=\frac{1}{3}$ state by addition of "second generation" Laughlin QHs. There, stability of the $\nu=\frac{2}{7}$ daughter does *not* require stability of the $\nu=\frac{1}{3}$ parent. We verified this fact directly by exact diagonalization of N electrons with the $e-e$ pseudopotential gradually weakened at short range. We found that breakup of Laughlin $\nu=\frac{1}{3}$ liquid precedes that of Jain $\nu=\frac{2}{7}$ daughter state.

The value of $2l=3N-7$ precludes a Laughlin state of pairs (or, equivalently, of the QHs). To show it, let us use the following pictorial argument, equivalent to a more rigorous derivation. Laughlin $\nu=\frac{1}{3}$ state (of individual particles) can be viewed as ●○○●⋯●○○●≡(●○○)●, with "●" and "○" denoting particles and vacancies. Counting the total LL degeneracy g leads to the correct value of $2l=3N-3$. The Moore-Read state, i.e., the Laughlin state of pairs at $\nu=\frac{1}{2}$, is represented by (●●○○)●●, yielding $2l=2N-3$. A

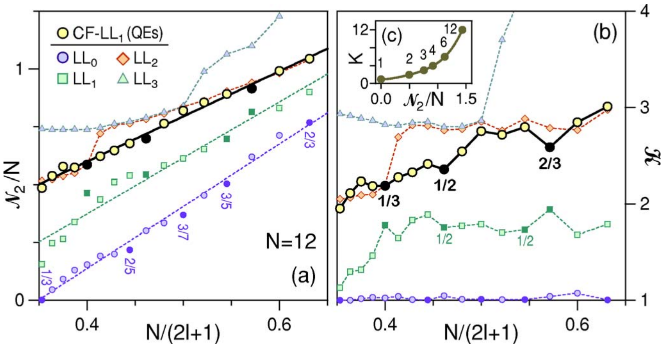

FIG. 4. (Color online) Number of pairs $\mathcal{N}_2$ with the minimum relative angular momentum $\mathcal{R}=1$ (a) and estimated average cluster size $\mathcal{K}$ (b) for $N=12$ electrons or composite fermions in Landau levels angular momentum shells with $2l=18$ to 33, plotted as a function of the filling factor $\nu \sim N/(2l+1)$.

Laughlin state of pairs at $\nu=\frac{1}{3}$ would correspond to (●●○○○○)●●, predicting (incorrectly) $2l=3N-5$. Assuming pairing, $2l=3N-7$ can only be obtained using a two-pair unit cell (●●○○●●○○○○○○○)●●○○●● corresponding to more complicated pair-pair correlations.

At higher fillings of CF-LL$_1$, $\mathcal{N}_3 \approx \frac{1}{3}N$ at $2l=20$ suggests division of N QEs into $\frac{1}{3}N$ triplets at $\nu=\frac{2}{3}$, and $\mathcal{N}_3 \approx \frac{1}{6}N$ at $2l=25$ implies a more complicated cluster configuration (with mixed sizes) at $\nu=\frac{1}{2}$. LL$_2$ and LL$_3$ look alike (and different from LL$_0$ or CF-LL$_1$) at $23 \le 2l < 29$, both having $\mathcal{N}_3 \approx \frac{1}{3}N$. At $2l=29$, $\mathcal{N}_3$ for LL$_2$ drops rapidly to almost zero. This further supports similarity of the $\nu=\frac{1}{3}$ states in LL$_2$ and CF-LL$_1$.

In Fig. 4(a) we replot $\mathcal{N}_2$ as a function of $N/g \sim \nu$. The quasilinear dependences for LL$_0$, LL$_1$, and CF-LL$_1$ all aim correctly at $\mathcal{N}_2=2N-3$ for $\nu=1$, but start from different values $\mathcal{N}_2 \approx 0$, $\frac{1}{4}N$, and $\frac{1}{2}N$, at $\nu=\frac{1}{3}$. Regular dependence allows subtraction from $\mathcal{N}_2$ the contribution from pairs belonging to different clusters. As a reference we used ground states of $V=\delta_{\mathcal{R},1}$. This short-range repulsion guarantees maximum avoidance of $\mathcal{R}=1$; its $\mathcal{N}_2^*$ contains only the intercluster contribution. To compare $\mathcal{N}_2$ of QEs or electrons with $\mathcal{N}_2^*$, we: (i) calculated $\mathcal{N}_2$ for a single K-size cluster, and multiplied it by N/K to obtain relation between $\mathcal{N}_2$ and K in an idealized clustered state of N particles, (ii) using this relation [see Fig. 4(c)], converted $\mathcal{N}_2$ and $\mathcal{N}_2^*$ into the (average) cluster sizes K and K^*; (iii) defined $\mathcal{K}=K-(K^*-1)$ as the cluster size estimate free of the intercluster contribution.

The result in Fig. 4(b) indicates pairing in LL$_1$ at $\frac{1}{3} \le \nu \le \frac{2}{3}$, and in both CF-LL$_1$ and LL$_2$ at $\nu \le \frac{1}{3}$. Triplets seem to form in CF-LL$_1$ at $\nu=\frac{2}{3}$, in LL$_2$ at $\frac{1}{3} \le \nu \le \frac{2}{3}$, and in LL$_3$ at $\nu \le \frac{1}{2}$. The $\nu=\frac{1}{2}$ state of QEs falls between $\mathcal{K}=2$ and 3, suggesting mixed-size clusters.

Finally, note that the concept of QE pairing at $\nu=\frac{1}{3}$, deduced here from the behavior of $\mathcal{N}_2/N$ and $\mathcal{N}_3/N$, might seem to contradict the earlier numerical studies[19] which showed that nondegenerate N-QE ground states with a gap occur at $2l=3N-7$ regardless of the parity of N. A possible explanation for this puzzle is that while a paired state is only possible for even N, the lowest-energy states at odd N will contain $\frac{1}{2}(N-1)$ pairs and one unpaired QE (assuming that pairing is indeed energetically favorable at this ν). This unpaired QE can be thought of as a defect in a liquid of pairs. In finite systems, its energy spectrum is quantized, possibly leading to a nondegenerate ground state (symmetric under defect hopping, i.e., pair-QE exchange). In small, numerically tractable systems, size quantization of the defect might easily exceed the tiny ($\sim 0.005e^2/\lambda$) incompressibility gap due the cluster-cluster interaction, causing the observed insensitivity of the total excitation gap to the parity of N. On the other hand, only the fully paired configurations would be relevant for very large systems.

In conclusion, we studied two- and three-body correlations of several quantum liquids, with correlated electrons or CFs in partially filled LLs. The most important result is for the relatively new fractional quantum Hall state at $\nu_e=\frac{4}{11}$, corresponding to the $\nu=\frac{1}{3}$ filling of the second CF LL and much less understood than, e.g., Laughlin, Jain, or Moore-Read states. We showed that (i) it is a paired liquid of CFs; (ii) it can be viewed as a $\nu=\frac{1}{2}$ MR state of CFs plus an appropriate number of its ("MR") QHs; (iii) since $\nu_e=\frac{4}{11}$ is known to be incompressible, the MR QHs must condense at this filling. Therefore, the $\nu_e=\frac{4}{11}$ state is interpreted as a condensate of "second generation" Moore-Read QHs.

Point (i) was proven directly by showing that the number of CF triplets vanishes and the number of CF pairs is half the number of CFs in the $\nu=\frac{1}{3}$ state of 12 CFs in the second CF LL. This demonstrates CF pairing at one-third filling in the second CF LL, reminiscent of Moore-Read electron pairing at one-half filling of the second electron LL. Conclusion (ii) follows from the facts that MR state at $\nu=\frac{1}{2}$ is paired and that its elementary excitations at $\nu<\frac{1}{2}$ are QHs and pair breakers. Since the $\nu=\frac{1}{3}$ state (of CFs in the second CF LL) has the same number of pairs as the MR state (of CFs in the second CF LL), it can be obtained from this MR state by addition of only QHs. The QH-QH interactions/correlations are unknown (except that the Laughlin form is excluded and that, nevertheless, the QHs condense), the identification of the role of QHs is important for their non-abelian statistics.

The tendency for pairing or clustering of CF's at sufficiently high ν was explained from their nonmonotonic charge densities, causing weak CF-CF repulsion at short range. However, the origin of incompressibility of CF pairs or larger clusters at precisely $\nu=\frac{1}{3}$, $\frac{1}{2}$, or $\frac{2}{3}$ is not yet clear, as its understanding requires better knowledge of the effective pair-pair or cluster-cluster interaction.

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