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Residual Interactions and Pairing of Composite Fermion Quasiparticles and Novel Condensed Quantum Fluid States

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The residual interactions between quasiparticles (QP's) in partially filled composite Fermion (CF) levels can be obtained from exact numerical diagonalization studies of small systems. The pseudopotentials $V_{\rm QP}(\mathcal{R})$ describing the energy of interaction of QE's (or QH's) as a function of their *relative angular momentum* $\mathcal{R}$ can be used to diagonalize the interactions among these CF QP's. It is known that $V_{\rm QP}(\mathcal{R})$ cannot support Laughlin correlations among CF QP's at $\nu_{\rm QE} = 1/3$ or $\nu_{\rm QH} = 1/5$. Because of this, the novel condensed quantum fluid states observed at $\nu = 4/11, 4/13$ and other Landau level filling fractions cannot possibly be spin-polarized Laughlin correlated QP states of the composite Fermion hierarchy. Formation of pairs or larger clusters of CF QP's clearly must occur, but the exact nature of the incompressible ground states is not completely understood.

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Fractional quantum Hall states have been observed recently at unexpected values of the electron Landau level (LL) filling factor ν [1]. Some of these states have been attributed to composite Fermions (CF's) of different "flavor" with the notations ^{2}CF, ^{4}CF, ... used for CF's with different numbers of attached Chern–Simons (CS) flux quanta [1,2]. This idea is equivalent to a CF hierarchy scheme [3], which involved the reapplication of the CS transformation to quasiparticles (QP's) in partially filled CF angular momentum shells or LL's proposed to

describe odd denominator fractions that did not belong to the Jain sequence [4] of filling factors. In addition, it is known that the reapplication of the CF transformation is valid only if the QP pseudopotential $V_{\rm QP}(\mathcal{R})$ supports Laughlin correlations [5].

By Laughlin correlations we mean the maximum avoidance of pair states with the largest pair angular momentum L' (or smallest value of the *relative angular momentum*, $\mathcal{R} = 2l - L'$, where l is the angular momentum of the individual particles). In order to support Laughlin correlations [5–7], the pseudopotential $V(L')$ describing the interaction energy of a pair of particles as a function of the pair angular momentum L' must increase, approaching the avoided value of L', more quickly than $L'(L'+1)$. We refer to such a potential as *superharmonic* since it increases more quickly than any $V_{\rm H}(L') = A + B\hat{L'}^2$ (where A and B are constants), defined as a *harmonic* pseudopotential [5–7].

The object of the present paper is to demonstrate that Laughlin correlations will not occur for the lowest energy states in the spectrum if the pseudopotential is subharmonic. We also show that, when $V_{\rm QP}(L')$ is not superharmonic, the interacting particles form pairs or larger clusters in order to lower the total energy [8]. These pairing correlations can lead to a nondegenerate incompressible ground state.

For electrons in the lowest LL ($n = 0$), pseudopotential $V_0(L')$ is superharmonic at every value of L'. For excited LL's ($n \geq 1$) $V_n(L')$ is not superharmonic at all the allowed values of L' [8]. In Fig. 1 we show the pseudopotentials of electrons in the lowest LL (a) and of quasielectrons (QE's) in the first excited CF LL (b). The QE pseudopotential is taken from the work of Lee *et al.* [9]. Neither is the pseudopotential $V_{\rm QP}(L')$ [5–10], describing the interaction of Laughlin QP's, superharmonic at all allowed values of L'.

The number of electrons required in order to have a system of QP pairs of reasonable size is, in general, too large for exact diagonalization in terms of electron states and the Coulomb pseudopotential [11]. However, by re-

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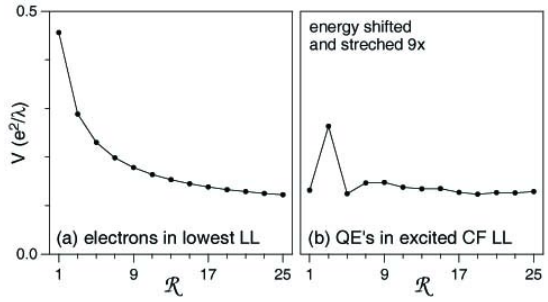

Fig. 1. Pseudopotentials (pair interaction energy V vs. relative pair angular momentum $\mathcal{R}$) for electrons in the lowest LL (a) and for QE's in the first excited CF LL (b). The values of V in frame (b) were calculated by Lee *et al.* [9] and are only known up to a constant. λ is the magnetic length.

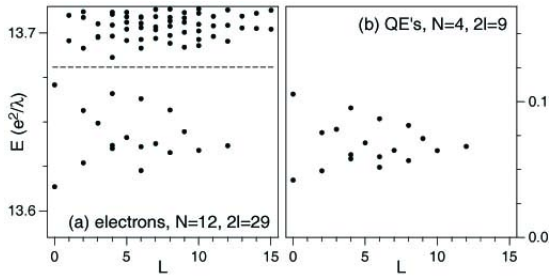

Fig. 2. Energy spectra for $N = 12$ electrons in the lowest LL with $2l = 29$ and for $N = 4$ QE's in the first excited CF LL with $2l = 9$. The energy scales are the same, but the QE spectrum obtained using $V_{\rm QE}(\mathcal{R})$ is determined only up to an arbitrary constant.

stricting our consideration to the QP's in the partially filled CF shell, and by using the QP pseudopotential obtained from numerical studies [5–10] of small systems of electrons, we can reduce the numerical diagonalization to manageable size [12]. The QP pseudopotentials determined in this way are quite accurate up to an overall constant which has no effect on the correlations. Furthermore, because the correlations are primarily determined by the short range part of the pseudopotential, the numerical results should describe the essential correlations quite well for systems whose size (e.g., Haldane sphere radius) is large compared to the correlation length.

In Fig. 2 we display the low-energy spectrum of $N = 12$ electrons at $2l = 29$, and the corresponding spectrum for $N_{\rm QE} = 4$ QE's at $2l_{\rm QE} = 9$ obtained in our *numerical experiments*.

[The CF transformation applied to the electrons gives an effective CF angular momentum $l^* = l - (N-1) = 7/2$. The lowest CF LL can accommodate $2l^* + 1 = 8$ of the particles, so that the first excited CF LL contains the remaining four QE's each of angular momentum $l_{\rm QE} = 9/2$]. The calculation for $2l_{\rm QE} = 9$ and $N_{\rm QE} = 4$ is almost trivial in comparison to that of $N = 12$ at $2l = 29$, but the low-energy spectra are in reasonably good agreement, giving us confidence in the use of $V_{\rm QP}(\mathcal{R})$ to

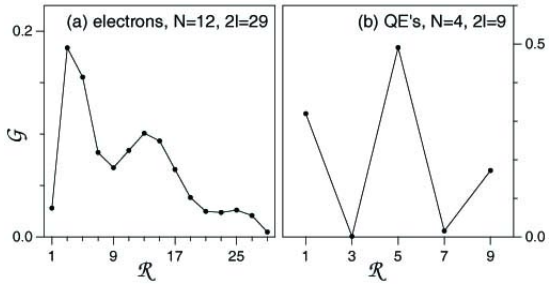

Fig. 3. Pair amplitude functions $\mathcal{G}(\mathcal{R})$ for the two ground states shown in Fig. 2.

describe the CF QP's.

In Fig. 3 we compare pair amplitude functions $\mathcal{G}(\mathcal{R})$ for the lowest $L = 0$ states appearing in Fig. 2. It is clear that the electrons are Laughlin correlated avoiding $\mathcal{R} = 1$ pair states, but the QE's are not since they avoid $\mathcal{R} = 3$ (and $\mathcal{R} = 7$) but not $\mathcal{R} = 1$.

The numerical diagonalizations of the interactions of the Fermions (electrons or CF QP's) within the Hilbert subspace of the partially occupied LL are performed in Haldane spherical geometry [13]. The particles are confined to a spherical surface of radius R, and a magnetic monopole of strength $2Q$ flux quanta at the center of the sphere produces a radial magnetic field $B = 2Q\phi_0/4\pi R^2$, where $\phi_0 = hc/e$. The single particle eigenfunctions in this geometry are called monopole harmonics and are denoted by $|Q, l, m\rangle$, where Q is half the monopole strength, l the angular momentum, and m its z-component. The single particle eigenvalues are given by $\epsilon_l = (\hbar\omega_c/2Q)\left[l(l+1) - Q^2\right]$, where ω_c is the cyclotron frequency. Because ϵ_l must be positive, the minimum value of l is Q, and we can label the angular momentum shells by $l_n = Q + n$, where n is a non-negative integer. For convenience of notation we will write the monopole harmonics as $|l, m\rangle$ with Q being understood.

For a system of N electrons confined to a shell of angular momentum l, we can form eigenfunctions with a given value of L, the total angular momentum, and M, its z-component. They can be written $|L, M, \alpha\rangle$ with the label α distinguishing distinct multiplets with the same values of L. The Wigner–Eckart theorem states for a scalar interaction H' that $\langle L', M', \alpha'| H' |L, M, \alpha\rangle = \delta_{LL'}\delta_{MM'} \langle L\alpha'| H' |L\alpha\rangle$ and that the reduced matrix element on the right hand side is independent of M. The eigenfunction for the α^{th} multiplet of total angular momentum L formed by adding the angular momenta $l_i = l$ of N identical Fermions can be written

$$\left|l^N; L\alpha\right\rangle = \sum_{L_{12}L''\alpha''} G_{L\alpha,L''\alpha''}(L_{12}) \left|l^2, L_{12}; l^{N-2}, L''\alpha''; L\right\rangle . \quad (1)$$

Here the $G_{L\alpha,L''\alpha''}(L_{12})$ are coefficients of fractional grandparentage [14]. The wavefunctions on the right hand side of Eq. (1) are obtained by adding the angular

momentum L_{12} of the pair $\langle 1,2\rangle$ to the angular momentum L'' of the α'' multiplet of the $j = 3, 4, \ldots, N$ remaining Fermions to obtain the total angular momentum L. Although $\left|l^2, L_{12}; l^{N-2}, L''\alpha''; L\right\rangle$ is not antisymmetric under the interchange of $i = 1$ or 2 with $j = 3, 4, \ldots, N$, the eigenfunctions $\left|l^N; L\alpha\right\rangle$ are totally antisymmetric. We define the pair amplitude $\mathcal{G}_{L\alpha}(L')$ by [15] $\mathcal{G}_{L\alpha}(L') = \sum_{L''\alpha''} |G_{L\alpha,L''\alpha''}(L')|^2$. Orthonormality of the eigenfunctions $\left|l^N; L\alpha\right\rangle$ gives the sum rule

$$\sum_{L'} \mathcal{G}_{L\alpha}(L') = 1. \tag{2}$$

A second useful sum rule

$$\frac{1}{2}N(N-1)\sum_{L'} L'(L'+1)\mathcal{G}_{L\alpha}(L') = L(L+1) + N(N-2)l(l+1) \tag{3}$$

can be obtained by using Eq. (1) together with the simple theorem on pair angular momenta $\hat{L}^2 + N(N-2)\hat{l}^2 = \sum_{\langle i,j\rangle}(\hat{l_i}+\hat{l_j})^2$ [7]. In this equation $\hat{l_i}+\hat{l_j}$ is the angular momentum operator of the pair $\langle i,j\rangle$, and the sum is over all pairs. The energy of the multiplet $|L\alpha\rangle$ is given by

$$E_\alpha(L) = \frac{1}{2}N(N-1)\sum_{L'} \mathcal{G}_{L\alpha}(L')V(L'), \tag{4}$$

where $V(L')$ is the pseudopotential. It is clear from Eq. (4) and the sum rules [Eqs. (2) and (3)] that, for a harmonic potential $V_{\rm H}(L')$, the energy is given by $E_\alpha(L) = c_1 + c_2 L(L+1)$ where c_1 and c_2 are independent of α. Because the right hand side of this equation is independent of α, every multiplet with the same value of L is degenerate, and the harmonic pseudopotential introduces no correlations [6,7]. Any linear combination of eigenfunctions with the same value of L (i.e., $\sum_\alpha c_\alpha |L\alpha\rangle$) has the same energy.

We can think of the pseudopotential $V(L')$ as a function of $\mathcal{R}$ since $\mathcal{R} = 2l - L'$, and write $V(\mathcal{R}) = V_{\rm H}(\mathcal{R}) + \Delta V(\mathcal{R})$. Correlations are completely determined by the anharmonic part $\Delta V(\mathcal{R})$. For a simple model in which $\Delta V = \triangle_1\delta_{\mathcal{R},1}$, with the constant $\triangle_1 > 0$, the lowest energy state for each value of L is the one with the smallest value of $\mathcal{G}_{L\alpha}(\mathcal{R} = 1)$, which we will call $\mathcal{G}_{L0}(\mathcal{R} = 1)$. This is exactly what we mean by Laughlin correlations. In fact, if $\triangle_1$ is infinite, the only states with finite energy are those for which $\mathcal{G}_{L0}(\mathcal{R} = 1)$ vanishes. The complete avoidance of the pair states with $\mathcal{R} = 1$ corresponds exactly to the Laughlin–Jastrow factor $\prod_{\langle i,j\rangle}(z_i - z_j)^2$ in the Laughlin wavefunction for the $\nu = 1/3$ state [16].

Now let's consider a model pseudopotential which can be superharmonic or subharmonic at $\mathcal{R} = 1$, viz., one in which $\Delta V(\mathcal{R}) = \triangle_1\delta_{\mathcal{R},1} + \triangle_3\delta_{\mathcal{R},3}$. We assert that if $\triangle_3$ is sufficiently large, Laughlin correlations will not produce the lowest state. We demonstrate this as follows: (i) the Laughlin correlated $L = 0$ ground state which occurs at $2Q = 3(N-1)$ when $\triangle_3 = 0$ must have the minimum possible value of $\mathcal{G}_0(\mathcal{R} = 1)$; (ii) in the presence of $\triangle_3 > 0$, decrease $\mathcal{G}_0(\mathcal{R} = 3)$ by an amount $\Delta\mathcal{G}$; (iii) in order to satisfy the first sum rule, Eq. (2), other pair amplitudes will have to increase.

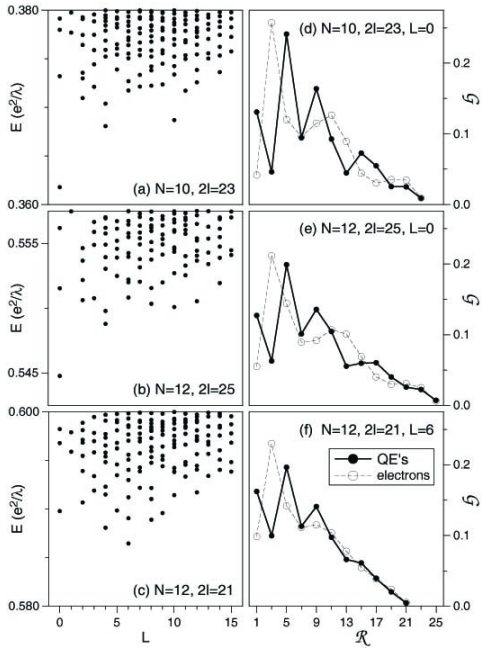

Fig. 4. Low energy spectra and pair amplitude functions: Frames (a), (b), and (c) show the energy spectra for $N = 10$ QE's at $2l = 23$, for $N = 12$ QE's at $2l = 25$, and for $N = 12$ QE's at $2l = 21$ as a function of total angular momentum L. Frames (d), (e), and (f) display pair amplitude functions $\mathcal{G}(\mathcal{R})$ for the ground states of the case presented in (a), (b), and (c), as a function of relative pair angular momentum $\mathcal{R}$. The solid circles are the ground state values of $\mathcal{G}(\mathcal{R})$ for the QE pseudopotentials. The open circles are the values for the superharmonic electron pseudopotential. All spectra were obtained using $V_{\rm QE}(\mathcal{R})$ given in Ref. [9].

For simplicity, let's assume that only $\mathcal{G}(\mathcal{R} = 1)$ and $\mathcal{G}(\mathcal{R} = j)$, with j an odd integer between $2l$ and 5, increase. By taking $\Delta\mathcal{G}(\mathcal{R} = 1) = x_j\Delta\mathcal{G}$ and $\Delta\mathcal{G}(\mathcal{R} = j) = (1 - x_j)\Delta\mathcal{G}$ along with $\Delta\mathcal{G}(\mathcal{R} = 3) = -\Delta\mathcal{G}$, the first sum rule is automatically satisfied. The second sum rule, Eq. (3), determines x_j, giving $x_j = 1 - 2(4l-3)(4l-j)^{-1}(j-1)^{-1}$. The change in energy of the $L = 0$ ground state in the presence of $\triangle_1$ and $\triangle_3$ is given by

$$\Delta E_0 = \Delta\mathcal{G}(x_j\triangle_1 - \triangle_3). \tag{5}$$

This becomes negative when $\triangle_3 > x_j\triangle_1$. For example, if we take $j = 5$, $x_5 = (4l-7)(8l-10)^{-1}$. The value of $\triangle_3 = x_5\triangle_1$ is exactly the same value that causes $\Delta V(\mathcal{R})$ to behave harmonically between $\mathcal{R} = 1$ and $\mathcal{R} = 5$. It always gives a superharmonic pseudopotential $V(\mathcal{R})$ at $\mathcal{R} = 3$, but at $\mathcal{R} = 1$, it is superharmonic only if $\triangle_3 < x_5\triangle_1$. It is not difficult to see that transfer of $\Delta\mathcal{G}$ to $\mathcal{R} = 1$ and $\mathcal{R} = 5$ results in the minimum value of x_j. The transfer of pair amplitude to pair states with $\mathcal{R} = 1$

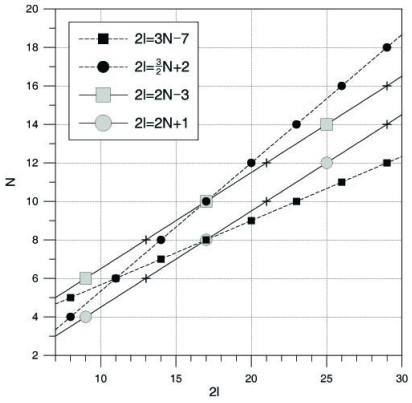

Fig. 5. The sequences of $\nu_{\rm QE} = 1/3$ (at $2l = 3N - 7$), $\nu_{\rm QE} = 2/3$ (at $2l = \frac{3}{2}N + 2$), and $\nu_{\rm QE} = 1/2$ (at $2l = 2N - 3$ and $2N + 1$) states shown as straight lines. The values of N and $2l$ at which $L = 0$ ground states separated from excited states by a substantial gap are shown as solid dots and solid squares (for $\nu_{\rm QE} = 1/3$ and $2/3$, respectively) and by open circles and open squares (for $\nu_{\rm QE} = 1/2$). The locations where $L = 0$ ground states of N QP's each with angular momentum l would be expected in the simple pairing model but are not found numerically are indicated by the symbol '+'.

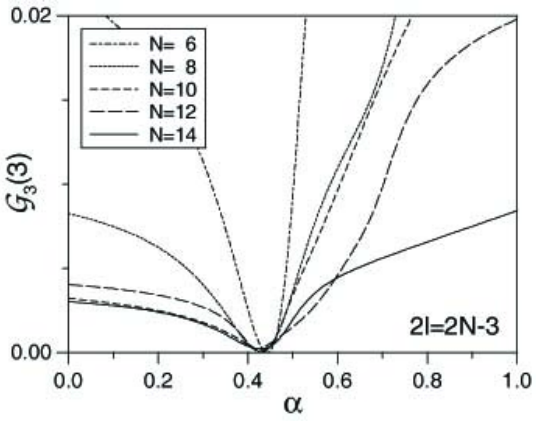

Fig. 6. Triplet amplitude $\mathcal{G}_3(\mathcal{R}_3 = 3)$ plotted as a function of α in the lowest $L = 0$ state of different numbers of Fermions N interacting through U_α in a shell with $2l = 2N - 3$.

together with the decrease in pair amplitude at $\mathcal{R} = 3$ is a clear indication of the *formation of Fermion pairs* with $\mathcal{R} = 1$ and the avoidance of pair states with $\mathcal{R} = 3$ (and the maximum repulsive interaction). Numerical studies [8, 17, 18] of small systems clearly support this picture when the pseudopotential is not superharmonic.

In Fig. 4, we present low energy spectra for three different cases: (a) is for $N = 10$ QE's at $2l = 23$, and corresponds to $\nu_{\rm QE} = 1/3$ and $\nu = 4/11$; (b) is for $N = 12$ QE's at $2l = 25$, and corresponds to $\nu_{\rm QE} = 1/2$ and $\nu = 3/8$; (c) is for $N = 12$ QE's at $2l = 21$, and it should also correspond to $\nu_{\rm QE} = 1/2$ and $\nu = 3/8$. The pseudopotentials given by Lee *et al.* [9] were used in obtaining these results. For small values of $\mathcal{R}$, their $V_{\rm QE}(\mathcal{R})$ agrees reasonably well with our earlier results [5–8,10], and the spectra and pair amplitudes are not very sensitive to which of these different $V_{\rm QE}(\mathcal{R})$ is used. The $\nu_{\rm QE} = 1/3$ state is one of a sequence of states occurring at $2l = 3N - 7$ whose spectra we have evaluated numerically for $4 \le N \le 12$. The other two states belong to the sequence $2l = 2N + 1$, which together with their conjugate states at $2l = 2N - 3$ (obtained by replacing N by $2l+1-N$, the number of QH's) correspond to $\nu_{\rm QE} = 1/2$ and $\nu = 3/8$. Frames (a) and (b) show $L = 0$ ground states separated by a substantial gap from excited states. Frame (c) does not have an $L = 0$ ground state, though a simple pairing model [17, 18] would predict one for this case. In frames (d), (e), and (f) the values of the pair amplitude functions $\mathcal{G}(\mathcal{R})$ for the ground states of (a), (b), and (c) are shown as solid dots. For the sake of contrast, $\mathcal{G}(\mathcal{R})$ for a superharmonic electron pseudopotential are shown as open circles. The pairing at $\mathcal{R} = 1$ and avoidance of $\mathcal{R} = 3$ QP states are quite clear.

A very simple pairing model based on Halperin's idea [15] was used [17,18] earlier which assumed that all the QE's formed $\mathcal{R} = 1$ pairs. The pairs can be treated as Fermions [17] or as Bosons [18] , and if Laughlin correlations between the pairs are assumed, incompressible ground states are formed at $\nu_{\rm QE} = 1/3$, $1/2$, and $2/3$ and $\nu_{\rm QH} = 1/5$, $1/4$, and $2/7$ giving novel condensed states at the values of LL fillings $\nu = 5/13$, $3/8$, $4/11$, and $\nu = 5/17$, $3/10$, $4/13$ observed experimentally [1]. However, the simple *complete pairing* model is probably too simple. Two major difficulties are not yet understood. First, the states obtained in our numerical calculations occur at $2l = 3N-7$ (for $\nu_{\rm QE} = 1/3$) for $N = 8$, 9, 10, 11, and 12, and at $2l = \frac{3}{2}N + 2$ (for $\nu_{\rm QE} = 2/3$) for $N = 10$, 12, 14, 16, and 18. *Complete pairing* can *only* occur for N even, and the sequence at $2l = 3N - 7$ occurs for both odd and even values of N. In addition, the simple *complete pairing* model would predict the $\nu_{\rm QE} = 1/3$ state at $2l = 3N - 5$ and the $\nu_{\rm QE} = 2/3$ state at $2l = \frac{3}{2}N + 1$, instead of at the values of $2l$ observed in the numerical study. Although this discrepancy is a finite size effect which becomes negligible for large N, we consider it important and are trying to understand its cause.

It is worth noting that the *formation of clusters* of k Fermions of angular momentum l (when the clusters themselves are treated as Fermions) results in condensed liquid states of Laughlin correlated clusters when $2l = mN - [(m - 1)k + 1]$. This would give correlated pair states at $2l = 2N - 3$ and correlated triplet states at $2l = 3N - 7$, as observed in our numerical results. Of course, the occurrence of complete triplet formation requires N to be divisible by 3, so it would only explain selected states in the $2l = 3N - 7$ sequence. We are still investigating what happens when incomplete clustering (simultaneously having single Fermions, Fermion pairs, Fermion triplets, etc.) occurs. The second problem is that the paired states at $2l = 2N - 3$ (and its e–h

conjugate states at $2l = 2N + 1$) do not occur at every expected even value of N in the *numerical experiments*.

Our numerical results [19] are summarized in Fig. 5, a plot of N versus $2l$ which contains four straight lines $2l = 3N - 7$, $2l = \frac{3}{2}N + 2$, $2l = 2N - 3$, and $2l = 2N + 1$. The last two are conjugate pair states for $\nu_{\rm QE} = 1/2$. The value at which $\nu_{\rm QE} = 1/3$ and $\nu_{\rm QE} = 2/3$ states found in our numerical experiments are shown as solid squares and solid dots, respectively. The values at which we find $\nu_{\rm QE} = 1/2$ states are shown as open circles and squares (the circles and squares surround the solid dots and solid squares at $2l = 17$, where $\nu_{\rm QE} = 1/2$ and $\nu_{\rm QE} = 1/3$ or $\nu_{\rm QE} = 2/3$ fit the observed states). The expected but unobserved states at $2l = 13$ (for $N = 6$ and 8), $2l = 21$ (for $N = 10$ and 12), and $2l = 29$ (for $N = 14$ and 16) are indicated by the symbol '+'.

We know [20] that for a model pseudopotential with $U_\alpha(\mathcal{R} = 1) = 1 - \alpha$ and $U_\alpha(\mathcal{R} = 3) = \alpha/2$, having approximately $\alpha \leq 0.25$ and $\alpha \geq 0.75$ leads to Laughlin correlations with $\mathcal{G}(\mathcal{R} = 3) \gg \mathcal{G}(\mathcal{R} = 1)$ and *anti*-Laughlin correlations with $\mathcal{G}(\mathcal{R} = 3) \ll \mathcal{G}(\mathcal{R} = 1)$, respectively. For $\alpha \approx 0.5$ (as in the first excited electron LL), $\mathcal{G}(\mathcal{R} = 3) \approx \mathcal{G}(\mathcal{R} = 1)$. For this case, the Moore–Read state [21] is considered a good description, and it is directly applicable to the $\nu = 5/2$ state which corresponds in the U_α to $\alpha \approx 0.5$.

A model three-body pseudopotential [22] $V_3(\mathcal{R}_3) = \delta_{\mathcal{R}_3,3}$ (where $\mathcal{R}_3 = 3l - L'$ and L' is the three-particle angular momentum) can be used to describe the Moore–Read correlations. In Fig. 6 we display $\mathcal{G}_3(\mathcal{R}_3 = 3)$, the amplitude for triplets with $\mathcal{R}_3 = 3$ (the smallest allowed value) as a function of α, the parameter in the two-body pseudopotential $U_\alpha(\mathcal{R})$.

It is clear that for $0.4 \leq \alpha \leq 0.5$, triplets with $\mathcal{R}_3 = 3$ are maximally avoided. However, for $\alpha \approx 1$, $\mathcal{G}_3(\mathcal{R}_3 = 3)$ is restored to a value even larger than that for $\alpha = 0$. This is certainly suggestive of clusters larger than pairs, and is currently being studied.

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