Proceedings of the 45th International School and Conference on the Physics of Semiconductors "Jaszowiec" 2016, Szczyrk

Quantum Hall State $\nu = 1/3$ and Antilexicographic Order of Partitions

B. Kuśmierz* and A. Wójs

Department of Theoretical Physics, Wrocław University of Science and Technology,
Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland

We focus on a certain aspect of trial wave function approach in the fractional quantum Hall effect. We analyze the role of partition orderings and discuss the possible numerical search for the partition determining the subspace of the Hilbert space containing a particular quantum Hall wave function. This research is inspired by analogical properties of certain polynomials which are the object of interest of the symmetric function theory, especially the Jack polynomials (related to the so-called "Jack states"). Presented method may be used in the search of candidate trial wave functions. We also justify (in certain cases) diagonalization of the Coulomb repulsion Hamiltonian restricted to certain subspaces. We focus on the states at filling factor $\nu = 1/3$ in the lowest and second Landau level.

DOI: 10.12693/APhysPolA.130.1183

PACS/topics: 73.43.–f, 71.10.Pm, 02.10.Ox

1. Introduction

Remarkable behavior of 2D electrons in low temperatures and strong magnetic field known as fractional quantum Hall effect (FQHE) is an actively studied area in condensed matter physics. Properties of essentially 2D electrons depend primarily on the Landau level (LL) filling factor ν (dimensionless ratio of electron and magnetic flux densities) [1–4]. Unlike in the case of integer quantum Hall effect where ν is a natural number, behavior of electrons at fractional ν cannot be explained without considering electron interaction. In fact, particles confined to the lowest Landau level (LLL) have the same kinetic energy and the total kinetic energy can be subtracted from the Hamiltonian as a constant leaving only interaction terms.

Among many approaches to FQHE there is one based on the symmetric function theory. In this approach proposed trial wave functions are developed by mathematicians and usually have a combinatorial interpretation. Most known family of symmetric polynomials that found applications in FQHE are Jack polynomials (Jacks) — J^{α}_{λ} [5–11]. FQH states related to Jacks are called "Jack states" and include: the Laughlin series, the Moore–Read "Pfaffian" state, the Read–Rezayi parafermion ground states and others [12, 13]. One of the important features of Jacks is that only certain coefficients of their expansion in the monomial basis are nonzero. Answer to the question which coefficients are nonzero is related to the structure of partitions and orderings of the partition space.

We analyze part of symmetric polynomials theory concerned with orderings of partitions in a context of physics of the Hall systems. Basis functions for lowest Landau level (LLL) quantum Hall states in both symmetric (monomial symmetric functions) and antisymmetric (the Slater determinants) are indexed by partitions. Mathematical study suggests that most natural way of ordering partitions is to do it according to so-called *natural order*. We examine whether such property can be of use in examination of fractional quantum Hall states.

The paper is organized as follows. In the following section a brief introduction to the symmetric function theory and partitions is given. In Sect. 3 FQH wave functions significant in our approach are discussed. In Sect. 4 results are presented.

2. Partitions and basis polynomials

The *partition* λ [5, 6] is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_j, \dots)$ of the non negative integers in non increasing order

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j \geq \dots \tag{1}$$

Partition can be an infinite sequence, however only a finite number of elements are nonzero. Usually when the partition indexes a polynomial, a sequence length corresponds to the number of variables of the polynomial. The nonzero λ_i are called the *parts* of λ, the number of parts is the *length* of λ and it is denoted by $\ell(\lambda)$. The sum of the parts of λ is called the *weight* and is denoted by $|\lambda| = \sum_i \lambda_i$. There is a natural addition of partitions

$$\lambda + \mu = (\lambda_1, \lambda_2, .., \lambda_j, ...) + (\mu_1, \mu_2, .., \mu_j, ...) = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, .., \lambda_j + \mu_2, ...). \tag{2}$$

Feature of partitions that we are especially interested in are partitions orderings. The *natural* ordering is defined as follows:

$$\lambda \geq \mu \Leftrightarrow \forall i : \lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i. \tag{3}$$

*corresponding author; e-mail: bartosz-kusmierz@o2.pl

Then one says λ *dominates* μ e.g. $(3,1) \geq (2,2)$. The natural order is not a *total order* and incomparable partitions occurs. For example $(4,1,1) \not\leq (3,3,0)$ and $(4,1,1) \not\geq (3,3,0)$. A total order consistent with natural order is the *reverse lexicographic order* [8]. One writes $\lambda \overset{R}{\geq} \mu$ when either $\lambda = \mu$ or the first non-vanishing difference $\lambda_i - \mu_i$ is positive. Considering the previous example $(4,1,1) \overset{R}{\geq} (3,3,0)$. Strict versions of mentioned inequalities occur when both compared partitions do not equal each other.

Space of quantum Hall wave functions in the lowest Landau level is spanned by polynomials, in the case of bosons (fermions) polynomials are symmetric (antisymmetric). Standard basis for symmetric polynomials are *monomial symmetric functions* (monomials) m_λ defined as:

$$m_\lambda(x_1, x_2, \ldots, x_N) = \\ F(\lambda) \sum_{\sigma \in S_n} x_1^{\lambda_{\sigma(1)}} x_2^{\lambda_{\sigma(2)}} \ldots x_N^{\lambda_{\sigma(N)}}. \quad (4)$$

For $F(\lambda)$ — the normalizing factor, given by

$$F(\lambda) = \frac{1}{m(\lambda,0)!m(\lambda,1)!\ldots}. \quad (5)$$

Similarly, basis of antisymmetric polynomials are *Slater determinants* sl_ν defined as:

$$\mathrm{sl}_\nu(x_1, x_2, \ldots, x_N) = \\ \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) x_{\sigma(1)}^{\nu_1} x_{\sigma(2)}^{\nu_2} \ldots x_{\sigma(N)}^{\nu_N}. \quad (6)$$

3. Partitions orderings and quantum Hall wave functions

Algebraic combinatorics brought to our attention that certain trial FQH wave functions are related to well known symmetric polynomials like Jack polynomials J_λ^α (parameter α is a real number, λ is a partition) [14–21]. Among those FQH states are the Laughlin series $\nu = 1/r$ (when r is even one gets bosonic states and for odd r fermionic states), the Moore–Read state $\nu = 1/2$ in the LL1, "parafermion" sequence $\nu = k/(k+2)$ and "Gaffnian" wave functions ($\nu = 2/3$ for bosons, $\nu = 2/5$ for fermions) [22, 23].

The Jack polynomials satisfy the following property that we call being "dominated" by partition: the only monomials contributing to the Jack polynomial are indexed by partitions dominated by λ

$$J_\lambda^\alpha = m_\lambda + \sum_{\mu < \lambda} v_{\lambda\mu}(\alpha) m_\mu. \quad (7)$$

Recursion formula for $v_{\lambda\mu}(\alpha)$ coefficients has been derived [11]. Property of being "dominated" by partition is common among symmetric polynomials like in the case of the Hall–Littlewood polynomials or the Macdonald polynomials $P_\lambda^{q,t}$ [5, 6, 24].

In our method we analyze FQH wave function obtained numerically by direct diagonalization of the Coulomb repulsion Hamiltonian. For a given numerically obtained function Ψ we want to find partition that "dominates" Ψ. We believe this is a first step to propose trial wave function approximating Ψ. Recognition of such partition would drastically diminish number of possible candidates of trial wave functions among known symmetric polynomials. For example checking all possible Jack polynomials J_λ^α as candidates for 10 particle Laughlin $\nu = 1/3$ wave function would require checking values of α for roughly 1.7×10^{11} partitions. Moreover, number of partitions that should be examined grows exponentially with number of particles. When "dominating" partition is fixed one can vary α parameter of Jack polynomials and consider overlaps with FQH function, then propose the one with highest overlap. Moreover, information that Ψ is contained in a subspace spanned only by certain polynomials imply that Hamiltonian can be diagonalized in smaller subspace (for example the Laughlin state $1/r$ is contained in the subspace spanned by polynomials dominated by $(Nr, (N-1)r, \ldots, r, 0)$).

To find "dominating" partition we order coefficients of wave function in such way that partitions indexing polynomials are ordered according to the reverse lexicographic order. For example expansion of the Laughlin wave function for three particles bosonic $\nu = 1/2$ can contain monomials indexed by partitions of weigth six and length not greater than three. Such partitions can be ordered: $(2,2,2) \overset{R}{\leq} (3,2,1) \overset{R}{\leq} (3,3) \overset{R}{\leq} (4,1,1) \overset{R}{\leq} (4,2,0) \overset{R}{\leq} (5,1,0) \overset{R}{\leq} (6,0,0)$. However not all monomials contribute

$$\Phi_{\mathrm{L}}^{1/2} = \prod_{1 \leq i < j \leq 3} (x_i - x_j)^2 = m_{(4,2,0)} - 2m_{(4,1,1)} \\ -2m_{(3,3)} + 2m_{(3,2,1)} - 6m_{(2,2,2)}.$$

Our goal is to find last partition with nonzero contribution to the wave function. In this case it would be $(4,2,0)$.

Results of our search of dominating partition for the state $\nu = 1/3$ are presented in a next section.

4. Results

We analyze spinless states of the Coulomb interacting electrons at $\nu = 1/3$ in LLL and first excited Landau level (LL1). Since system is fermionic we consider antisymmetric polynomials with the Slater determinants basis. Considered LL1 state corresponds to the physical filling factor $\nu = 7/3$. Also we vary number of particles in a search of partition that "dominates" given wave functions.

Let Ψ be a numerically obtained FQH wave function. Denote the dimensionality of the Hilbert space as dim, then there are exactly dim Slater determinants spanning this Hilbert space. Now order partitions indexing those Slater determinants according to the reverse lexicographic order: $\mu^{(1)} \overset{R}{<} \mu^{(2)} \overset{R}{<} \ldots \overset{R}{<} \mu^{(\dim)}$. For a partition $\mu^{(i)}$ we call i its coordination number. Let $\Psi_{\mu^{(i)}}$ be a scalar product of Ψ and basis polynomials indexed

by partition $\mu^{(i)}$ (coefficients of Ψ in basis). In order to find "dominating" partition we consider the following function:

$$F(i) = \frac{\sum_{j\leq i} |\Psi_{\mu^{(j)}}|}{i} - \frac{\sum_{j>i} |\Psi_{\mu^{(j)}}|}{\dim - i}. \quad (8)$$

For a given i the expression is a difference of average absolute values of coefficients indexed by partitions smaller or equal to $\mu^{(i)}$ and average absolute value of coefficients bigger in reverse lexicographic order.

Assume partition "dominating" is $\mu^{(k)}$, then for $l > k$ $\Psi_{\mu^{(l)}}$ should equal zero or be within range of numerical error. Then

$$\frac{1}{l}\sum_{j\leq l} |\Psi_{\mu^{(j)}}| < \frac{1}{k}\sum_{j\leq k} |\Psi_{\mu^{(j)}}|$$

as left-hand side is an average of the same elements as right-hand side and $(l-k)$ zeros. On the other hand, $(\dim - l)^{-1}\sum_{j>l} |\Psi_{\mu^{(j)}}| \approx (\dim - k)^{-1}\sum_{j>k} |\Psi_{\mu^{(j)}}|$ as both expressions are averages over elements close to zero. Thus inequality $F(k) > F(l)$ should occur. Similarly for $l < k$ first term (minuend) should be the same for both l and k as it is an average over many elements, but second term (subtrahend) should increase as we add positive numbers to the sums of numbers close to zero. Thus once more $F(k) > F(l)$ should occur. Thus we search for the maximum in $F(i)$.

TABLE I

Partitions "dominating" FQH ground state wave functions for $\nu = 1/3$ within LLL and LL1 obtained by searching for maximum of $F(i)$. Consecutive columns are: Landau level, electron number N, dimension of the relevant Hilbert space, "dominating partition", coordination number of partition and sum of squares of the coefficients indexed by partitions smaller than "dominating" which correspond to amount of wave functions confined to the subspace spanned by mentioned polynomials. The Coulomb states have been obtained numerically for the layer with zero width.

N	dim	Partition	Coordination number	Ψ in spanned subspace
		LLL		
4	18	$(9,6,3,0)$	16	0.9991
5	73	$(12,9,6,3,0)$	62	0.9995
6	338	$(15,12,9,6,3,0)$	279	0.9999
7	1656	$(18,15,12,9,6,3,0)$	1348	0.9990
		LL1		
4	18	$(8,5,4,1)$	7	0.4825
5	73	$(12,11,5,4,1)$	72	0.9997
6	338	$(15,14,11,4,1,0)$	336	0.9999
7	1656	$(18,17,16,8,3,1,0)$	1655	0.9999

Overview of Table I and Figs. 1, 2 reveals that dominating partitions for the Coulomb repulsion state $\nu = 1/3$ in LLL seems to be following known pattern of $(3(N-1), 3(N-2), \ldots, 6, 3, 0)$. It should be stated that the Laughlin $\nu = 1/3$ wave function follows exactly the same pattern, as there is a Jack polynomial indexed by this partition that equals the Laughlin wave function. Even though partition $(3(N-1), 3(N-2), \ldots, 6, 3, 0)$ is far from being the biggest in reverse lexicographic order (see Fig. 1) most of the wave function is contained within the Hilbert subspace spanned by polynomials indexed by partitions smaller that "dominating" partition (more than 0.99). Data for LL1 suggest that there is no candidate for partition "dominating" this wave function as partitions from Table I do not follow any clear pattern. Moreover unlike in the case of LLL maximum of $F(i)$ is reached for i very close to dim (see Fig. 2). Thus we conclude that there is little probability of finding trial function describing this state among polynomials with "dominating" partition like the Jack or Macdonald polynomials.

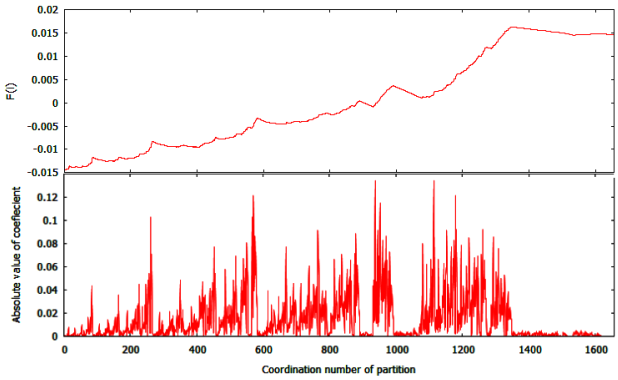

Fig. 1. Values of $F(i)$ (top) and absolute values of $\Psi_{\mu^{(i)}}(i)$ (bottom) for seven electron wave function of Coulomb repulsion ground state in LLL. One can notice that maximum of $F(i)$ in the top picture coincidences with part of bottom picture where overlap of the wave function with a polynomial indexed by partition coordination number starts to take very small values.

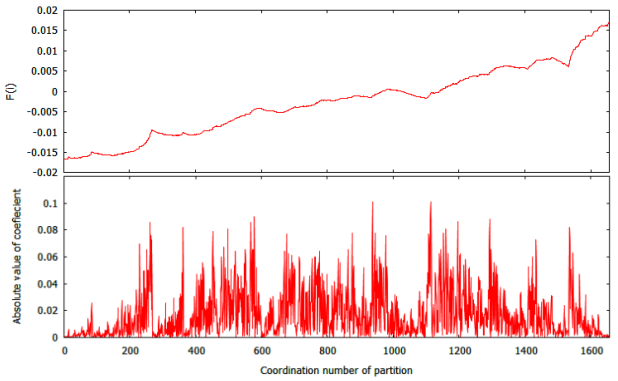

Fig. 2. Values of $F(i)$ (top) and absolute values of $\Psi_{\mu^{(i)}}(i)$ (bottom) for seven electron wave function of Coulomb repulsion ground state in LL1.

Acknowledgments

We acknowledge financial support from the Polish NCN Grant No. 2014/14/A/ST3/00654.

References

[1] J.K. Jain, *Composite Fermions*, Pennsylvania State University, 2012.

[2] J.K. Jain, *Phys. Rev. Lett.* **63**, 199 (1989).

[3] R. Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983).

[4] D.C. Tsui, H.L. Störmer, A.C. Gossard, *Phys. Rev. Lett.* **48**, 1559 (1982).

[5] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press, New York 1995.

[6] I.G. Macdonald, *A New Class of Symmetric Functions*, Publ. I.R.M.A. Strasbourg, 372/S20, Actes 20 Séminaire Lotharingien, 1988, p. 131.

[7] S. Kerov, *Asymptotic Representation Theory of the Symmetric Group and its Applications in Analysis*, American Mathematical Society, United States of America, 2003.

[8] R.P. Stanley, *Adv. Math.* **77**, 76 (1988).

[9] F.K. Sogo, *J. Math. Phys.* **35**, 2282 (1994).

[10] A. Hora, N. Obata, *Quantum Probability and Spectral Analysis of Graphs*, Springer-Verlag, Berlin 2007.

[11] L. Lapointe, A. Lascoux, J. Morse, *Elec. J. Combin.* **7**, N1 (2000).

[12] G. Moore, N. Read, *Nucl. Phys. B* **360**, 362 (1991).

[13] N. Read, E. Rezayi, *Phys. Rev. B* **59**, 8084 (1999).

[14] B.A. Bernevig, F.D.M. Haldane, *Phys. Rev. B* **77**, 184502 (2008).

[15] B.A. Bernevig, F.D.M. Haldane, *Phys. Rev. Lett.* **100**, 246802 (2008).

[16] B.A. Bernevig, F.D.M. Haldane, *Phys. Rev. Lett.* **102**, 066802 (2009).

[17] A. Bernevig, N. Regnault, *Phys. Rev. Lett.* **103**, 206801 (2009).

[18] R. Thomale, B. Estienne, N. Regnault, A. Bernevig, *Phys. Rev. B* **84**, 045127 (2011).

[19] W. Baratta, P.J. Forrester, *Nucl. Phys. B* **843**, 362 (2011).

[20] B. Kuśmierz, Y.-H. Wu, A. Wójs, *Acta Phys. Pol. A* **126**, 1134 (2014).

[21] B. Kuśmierz, Y.-H. Wu, A. Wójs, *Acta Phys. Pol. A* **129**, A-73 (2016).

[22] K.H. Lee, Z.-X. Hu, X. Wan, *Phys. Rev. B* **89**, 165124 (2014).

[23] M.V. Milovanovic, Th. Jolicoeur, I. Vidanovic, *Phys. Rev. B* **80**, 155324 (2009).

[24] D.E. Littlewood, *Proc. Lond. Math. Soc.* **43**, 485 (1961).