Possible Anti-Pfaffian Pairing of Composite Fermions at $\nu = 3/8$

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(Received 8 July 2012; published 18 December 2012)

We predict that an incompressible fractional quantum Hall state is likely to form at $\nu = 3/8$ as a result of a chiral p-wave pairing of fully spin polarized composite fermions carrying four quantized vortices, and that the pairing is of the anti-Pfaffian kind. Experimental ramifications include quasiparticles with non-Abelian braid statistics and upstream neutral edge modes.

DOI: 10.1103/PhysRevLett.109.256801 PACS numbers: 73.43.−f, 05.30.Pr, 71.10.Pm

Many novel structures and ideas arising in the study of quantum Hall effect, a topological state of matter, have generated new developments in other contexts, for example, topological insulators, Chern insulators, Majorana fermions, and quantum computation. The topological nature of the fractional quantum Hall effect (FQHE) manifests through formation of composite fermions (CFs), through Abelian and non-Abelian braid statistics, and also through the structure of the gapless edge modes. A focus of recent attention has been the $5/2$ state, believed to be a chiral p-wave paired state of composite fermions supporting Majorana zero modes obeying non-Abelian braid statistics [1–5]. In this Letter we propose that the mechanism of CF pairing is likely to produce an incompressible FQHE state also at filling factor $3/8$ in the lowest Landau level (LL) and enumerate many experimentally testable consequences arising from this physics, including non-Abelian braid statistics and the presence of upstream neutral edge modes. The possibility of CF pairing at $3/8$ was considered theoretically in several previous articles [6–8], which are discussed at the end in the context of the present work. While FQHE at $3/8$ has not been established conclusively, experimental indications for it have been seen by Pan *et al.* [9] and Bellani *et al.* [10].

Our calculations below demonstrate that for fully polarized electrons, the $3/8$ state is accurately described as the $\nu^* = 3/2$ state of composite fermions carrying two vortices (^{2}CFs), and thus represents a ^{2}CF analog of the half filled second LL. We provide evidence that the composite fermions in the half filled second CF LL (called ΛL) capture two additional vortices to turn into higher order composite fermions (^{4}CFs) and condense into a paired FQHE state. For the $5/2$ FQHE there are two topologically distinct candidates for the paired CF state: the Pfaffian (Pf) [1] and its hole partner known as the anti-Pfaffian (APf) [11–13]; a 3-body interaction term induced by LL mixing breaks particle hole symmetry and selects one of these states [14,15]. Two candidate states are obtained also at $3/8$ by composite fermionizing the Pf and APf at $3/2$. Our calculations show that the Coulomb interaction favors the APf state at $3/8$. Interestingly, LL mixing is not necessary for discriminating between the Pf and the APf at $3/8$—the Coulomb interaction between electrons induces a complex effective interaction between composite fermions that automatically contains 2-, 3-, and higher body terms.

The gap at $3/8$, and the difference between the Pf and the APf, is governed by extremely small energy scales, and a theoretical resolution of these states requires a precise and reliable quantitative treatment of the inter-CF interaction. We will consider N electrons moving on the surface of a sphere, subjected to a net magnetic flux of $2Q$ flux quanta [16]. We will assume that the spin degree is frozen, and the magnetic field is high enough that LL mixing is suppressed. The filling factor is defined as $\nu = \lim_{N\to\infty} \frac{N}{2Q}$. Composite fermions [17] experience an effective flux $2Q^* = 2Q - 2(N-1)$. At half filled second ΛL, the composite fermions satisfy $2Q^* + 2 = 2N_2 + \lambda$, where N_2 is the number of composite fermions in the second ΛL and λ is an integer "shift." This leads to the following relations at $\nu = 3/8$:

$$2Q = \frac{8N + \lambda - 10}{3}, \qquad 2Q^* = \frac{2N + \lambda - 4}{3}, \qquad N_2 = \frac{N - \lambda + 1}{3}.$$

We refer to $2Q$ given by the above relation as the "Pf flux" for $\lambda = -3$ and the "APf flux" for $\lambda = 1$. (At this stage, these terms should be taken only as convenient labels and not to mean that the actual states at these fluxes are represented by the Pf and the APf wave functions.)

Exact diagonalization is possible for 14 (12) electrons at Pf (APf) flux, but not for larger systems [18]. Further progress, however, can be made within the CF theory. We determine the energies and wave functions for low lying states by the method of CF diagonalization (CFD) [19], which proceeds along the following steps. We first perform exact diagonalization of the Coulomb Hamiltonian at Q^* ($\nu^* = 3/2$) keeping the lowest LL fully occupied, to obtain a basis $\{\Phi^{L,\alpha}_{3/2}\}$, where α labels the different basis functions in the total angular momentum L sector. (Which interaction is chosen is unimportant because our goal is to produce all basis states with the lowest kinetic energy.) We then composite fermionize this basis through the relation $\Psi^{L,\alpha}_{3/8} = P_{\rm LLL}\prod_{j<k}(u_j v_k - v_j u_k)^2 \Phi^{L,\alpha}_{3/2}$, where $u = \cos(\theta/2)e^{-i\phi/2}$, $v = \sin(\theta/2)e^{i\phi/2}$, and $P_{\rm LLL}$ denotes projection of the wave function into the lowest LL, handled by the method in Ref. [20]. The correlated states $\{\Psi^{L,\alpha}_{3/8}\}$ give us a basis for the low energy CF states at $\nu = 3/8$. All these states would be degenerate if composite fermions were noninteracting, but the degeneracy between them is split because of the residual interaction between composite fermions. We determine the low energy spectrum by diagonalizing the full Coulomb Hamiltonian in the CF basis (which can be performed in each L sector separately). The basis functions are very complex and nonorthogonal, but efficient methods have been developed for a Gram-Schmid orthogonalization and an evaluation of the Hamiltonian matrix by Metropolis Monte Carlo calculations [19]. A diagonalization of this matrix produces the low energy spectra as well as eigenfunctions. These contain no adjustable parameters, and the Monte Carlo statistical uncertainty can be reduced to the desired level by increasing the number of iterations accordingly (which, for our calculations, requires up to 10^8 Monte Carlo steps for each system). We study systems with as many as 26 particles, which allows us to draw what we believe to be reliable conclusions.

In Fig. 1 we compare the CFD spectra with those obtained from an exact diagonalization of the Coulomb interaction in the full lowest LL space for $N = 14$ at the Pf flux and $N = 12$ at the APf flux. These comparisons show that (i) the physics of the 3/8 state is indeed described in terms of composite fermions, and (ii) the CFD gives an essentially exact account of the inter-CF interaction. It is important for our purposes to note that the CF spectra not only reproduce the exact Coulomb spectra accurately, but also capture the very slight differences between the Coulomb spectra at the Pf and the APf fluxes. (The presence of such differences indicates that the particle hole symmetry is not exact for composite fermions.)

CF spectra for larger systems are shown in Fig. 2. A necessary condition for incompressibility is a spatially uniform $L = 0$ ground state. The fact that all of the APf flux values produce $L = 0$ ground states (but not all of the Pf flux values do) suggests that an incompressible state occurs at 3/8 at the APf flux. The system sizes are still not large enough to be able to estimate the gap reliably, but we note that the gap to the lowest neutral excitation for the two largest systems is $\sim 0.002e^2/\epsilon l$, which we take as a measure of the energy scale associated with this state. For a given density, this is roughly a factor of 5 smaller than the theoretical gap of the 5/2 state ($\sim 0.028e^2/\epsilon l$ [21]), taking into account the different magnetic lengths at the two fractions.

For a further confirmation that the actual state is indeed described by the APf wave function, we construct the

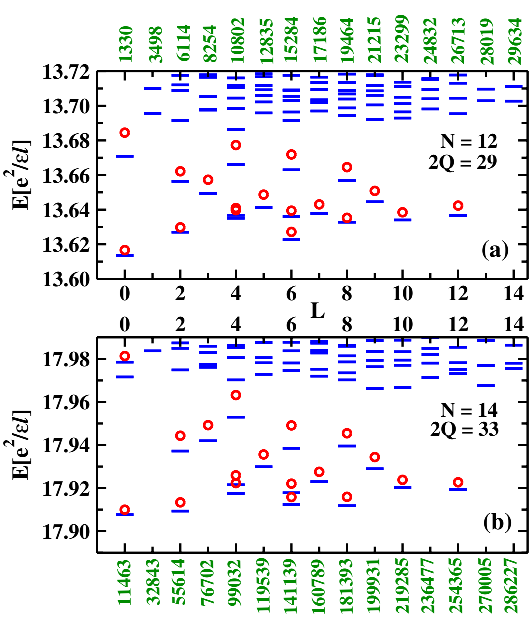

FIG. 1 (color online). Exact Coulomb spectra (dashes) at $[N, 2Q] = [12, 29]$ and [14,33], which correspond to APf and Pf fluxes at 3/8. Spectra obtained from composite fermion diagonalization are also shown (circles). The energies here and in Fig. 2 are the total Coulomb energies, which do not include the neutralizing background. The dimensions of the Hilbert space in the individual L sectors are shown at the top and the bottom.

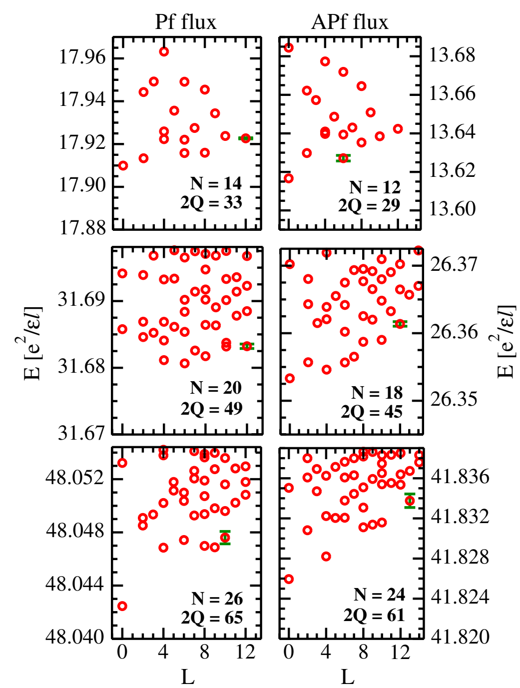

FIG. 2 (color online). Energy spectra obtained from CF diagonalization at both Pf flux (left-hand panels) and APf flux (right-hand panels) at $\nu = 3/8$, with N and $2Q$ values shown on the figure. To avoid clutter, the typical estimated statistical uncertainty from Metropolis Monte Carlo evaluation of integrals is shown only on one point. Only states below certain energy are shown.

following trial wave functions, labeled 1 and 2, at the Pf and the APf flux values:

$$\Psi_{3/8}^{\text{trial-1}} = P_{\text{LLL}} \prod_{j<k} (u_j v_k - v_j u_k)^2 \Phi_{3/2}^{\text{Pf/APf}},$$

$$\Psi_{3/8}^{\text{trial-2}} = P_{\text{LLL}} \prod_{j<k} (u_j v_k - v_j u_k)^2 \Phi_{3/2}^{\text{Coulomb}}.$$

Here, $\Phi_{3/2}^{\text{Pf/APf}}$ is the Pf or APf wave function at 3/2, which refers to the state in which the lowest LL is fully occupied and the electrons in the second LL form a Pf or an APf state. [We produce the Pf state in the lowest LL by diagonalizing the 3-body interaction Hamiltonian [2] $V_3 = \sum_{i<j<k} P_{ijk}^{(3)}(3Q-3)$, where $P_{ijk}^{(3)}(L)$ projects the state of the three particles (i, j, k) into the subspace of total orbital angular momentum L; the APf state is obtained by its particle hole conjugation. We then elevate the Pf/APf to the second LL and fill the lowest LL fully to obtain $\Phi_{3/2}^{\text{Pf/APf}}$.] The wave function $\Phi_{3/2}^{\text{Coulomb}}$ is the exact Coulomb eigenstate at the relevant Q^* at $\nu^* = 3/2$. Composite fermionization of these wave functions gives two trial wave functions at 3/8. Tables I and II compare the energies of these trial wave functions with the CFD energies, and also give the overlaps of these trial wave functions with the CFD wave function. The APf state has higher overlaps, again indicating that it is favored over the Pf. The overlaps are not extremely high, but on the same order as the overlaps of the 5/2 Coulomb ground state with the Pf/APf wave function. Taking into account these facts, we conclude that it is likely that the 3/8 state is incompressible and described by a composite-fermionized APf state.

The principal consequences arising from our calculations above are that (i) FQHE is possible at 3/8 (Pan *et al.* had observed [9] a resistance minimum at 3/8, but a well quantized plateau has not been seen so far), (ii) it originates due to p-wave pairing of composite fermions in the second Λ level, and (iii) the pairing is of the APf type. We obviously cannot rule out that the system sizes considered here may not capture the true nature of the thermodynamic phase, and the eventual confirmation will likely come from experiments. We now list some experimental consequences of the above physics. (i) The 3/8 FQHE state should be fully spin polarized (as is also the case for the 5/2 state [22]). (ii) The chiral p-wave pairing reflects through the charge and non-Abelian braid statistics of the quasiparticles ("composite non-Abelians"). The excess charge associated with an excitation is $e/16$, and its braid statistics will have similar signatures as those predicted for 5/2 [23,24]. (iii) Proposals have been made for experimentally distinguishing the Pf and the APf states at 5/2 through their different edge structures [11,12,25,26], and these analyses carry over to the 3/8 state with appropriate modifications. The Pf and APf states at 3/2 have edge structures (disregarding the possibility of edge reconstruction) 3/2(Pf)-1-0 and 3/2(APf)-2-1-0, respectively, which translate, upon composite fermionization, into 3/8(Pf)-1/3-0 and 3/8(APf)-2/5-1/3-0 at 3/8. An immediate consequence is that the APf will *necessarily* contain counterpropagating edge modes, including an upstream charge neutral Majorana mode, which can have experimental signatures, e.g., in noise measurements in an upstream voltage contact [27]. Observation of such modes would not constitute a proof of APf, because the Pf state can also have backward moving modes due to edge reconstruction. However, we expect that the physics of edge reconstruction at 3/8 should not be too different from that at the nearby fractions 1/3 or 2/5, so an observation of counterpropagating modes at 3/8 concurrent with an absence of such modes at 1/3 and 2/5 can be taken as a substantial evidence for APf state at 3/8. The thermal Hall conductivity $K_H = \partial J_Q/\partial T$, where J_Q is the thermal energy current and ∂T is the "Hall" temperature difference, can also in principle distinguish between the Pf and the APf [11]. In units of $(\pi^2 k_B^2/3h)T$, each chiral boson edge mode contributes one unit and the Majorana fermion mode 1/2 unit [28,29], with the sign depending on the direction of propagation. The boundary 3/8(Pf)-1/3 supports a chiral boson and a Majorana mode; the boundary 3/8(APf)-2/5 also supports a chiral boson and a Majorana mode, but moving in the upstream direction. This produces thermal Hall

TABLE I. Comparing the CFD ground state $\Psi_{3/8}^{\text{CFD}}$ at the Pf flux $2Q = (8N-13)/3$, obtained by CF diagonalization, with the trial wave functions, $\Psi_{3/8}^{\text{trial-1}}$ and $\Psi_{3/8}^{\text{trial-2}}$, derived from the composite fermionization of the Pf and the exact Coulomb states at 3/2. (See text for definition.) $E_{3/8}^{\text{CFD}}$, $E_{3/8}^{\text{trial-1}}$, and $E_{3/8}^{\text{trial-2}}$ are the energies per particle for these three states, quoted in units of $e^2/\epsilon l$, where $l = \sqrt{\hbar c/eB}$ is the magnetic length and ϵ is the dielectric constant of the background material; this energy includes the interaction with the positively charged background. The numbers $O_j = \langle \Psi_{3/8}^{\text{trial-}j} | \Psi_{3/8}^{\text{CFD}} \rangle$ are the overlaps of the two trial wave functions with the CFD ground state (all properly normalized). The asterisk indicates that for $N = 20$ the comparisons are given for the lowest energy state in the $L = 0$ sector; the CFD ground state occurs at $L = 6$.

N	O_1	O_2	$E_{3/8}^{\text{trial-1}}$	$E_{3/8}^{\text{trial-2}}$	$E_{3/8}^{\text{CFD}}$
14	0.726(1)	0.973(2)	−0.44153(8)	−0.44372(9)	−0.44403(9)
20*	0.379(1)	0.434(1)	−0.43418(2)	−0.43515(8)	−0.43599(1)
26	0.271(1)	0.526(1)	−0.43021(9)	−0.43146(6)	−0.43248(4)

TABLE II. Comparing the CFD state at APf flux $2Q = (8N-9)/3$ with two trial wave functions, $\Psi_{3/8}^{\text{trial-1}}$ and $\Psi_{3/8}^{\text{trial-2}}$, obtained by composite fermionization of the APf and the exact Coulomb states at 3/2. Other symbols have the same meaning as in Table I.

N	O_1	O_2	$E_{3/8}^{\text{trial-1}}$	$E_{3/8}^{\text{trial-2}}$	$E_{3/8}^{\text{CFD}}$
12	0.816(1)	0.994(1)	−0.43903(2)	−0.44076(6)	−0.44079(9)
18	0.587(2)	0.622(2)	−0.43168(9)	−0.43225(7)	−0.43310(8)
24	0.503(1)	0.781(1)	−0.42845(9)	−0.42948(8)	−0.42995(7)

conductivity of $1 + 1/2 + 1 = 5/2$ for the Pf and $-1 - 1/2 + 1 + 1 = 1/2$ for the APf at 3/8. This result is believed to be robust against interactions, disorder, or edge reconstruction. One may also consider various tunneling exponents, following Wen [25,26]. The exponent describing the long distance decay of the propagator of the charge 1/16 non-Abelian quasiparticles can be shown [30] to be $g = 7/13$ for the 3/8 Pf; this exponent appears in the prediction [26], assuming absence of edge reconstruction, that the current from one edge of the sample to the opposite edge near a quantum point contact satisfies $I \sim V^{2g-1}$ and the tunnel conductance has a temperature dependence $\sigma \sim T^{2g-2}$. For the APf state, on the other hand, the presence of upstream neutral modes renders the various exponents nonuniversal even for an unreconstructed edge.

The earlier studies of the $\nu = 3/8$ considered composite fermions interacting with a 2-body interaction, the form of which is determined [6,14,31] by considering two composite fermions in the second ΛL. This method is less accurate than the CFD used above, and, in particular, cannot discriminate between the Pf and the APf states because, by construction, it obeys particle-hole symmetry for composite fermions. Reference [6] evaluated the energies of variational wave functions for the Pf, stripe, Wigner crystal, and Fermi sea states at 3/8, and concluded that the stripe phase has the lowest energy; the conclusion, however, rests sensitively on the quality of various trial wave functions used in the study. Reference [8] investigated the 3/8 sate by a numerical diagonalization of the same 2-body model interaction, but did not find incompressible states at all even N. Reference [7] considered composite fermions in the spin reversed $n = 0$ ΛL, also using a 2-body interaction model for composite fermions, and pointed toward a partially spin polarized paired FQHE state; such a state is unlikely to be relevant at very high magnetic fields, e.g., in Ref. [9]. We finally note that we have not included in our work the effect of finite thickness, LL mixing, and disorder; while these will surely make quantitative corrections, we do not see any reason why they should change the qualitative physics of the state.

We thank Michael Levin for a valuable discussion, Parsa Bonderson for a useful comment, and acknowledge financial support from the NSF under Grant No. DMR-1005536, the DOE under Grant No. DE-SC0005042 (J. K. J.), the Polish NCN Grant No. 2011/01/B/ST3/04504, EU Marie Curie Grant No. PCIG09-GA-2011-294186 (A. W.), and CSIR, Government of India (S. M.). The computations were performed using the cluster of the Department of Theoretical Physics, IACS, and the Wroclaw Centre for Networking and Supercomputing.

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