Pair-distribution functions of correlated composite fermions

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(Received 2 February 2005; revised manuscript received 4 May 2005; published 30 June 2005)

Pair-distribution functions $g(r)$ of Laughlin quasielectrons (composite fermions in their second Landau level) are calculated in the fractional quantum Hall states at electron filling factors $\nu_e=4/11$ and $3/8$. A shoulder in $g(r)$ is found, supporting the idea of cluster formation. The intra- and intercluster contributions to $g(r)$ are identified, largely independent of ν_e. The average cluster sizes are estimated; pairs and triplets of quasielectrons are suggested at $\nu_e=4/11$ and $3/8$, respectively.

DOI: 10.1103/PhysRevB.71.245331 PACS number(s): 73.43.−f, 71.10.Pm

I. INTRODUCTION

Pan *et al.*[1] have recently observed the fractional quantum Hall effect[2,3] (FQHE) in a spin-polarized two-dimensional electron gas (2DEG) at the $\nu_e=\frac{4}{11}$, $\frac{3}{8}$, and $\frac{5}{13}$ fillings of the lowest Landau level (LL). In the composite fermion (CF) model,[4,5] these values correspond to the fractional fillings $\nu=\frac{1}{3}$, $\frac{1}{2}$, and $\frac{2}{3}$ of the second CF LL, respectively. In Haldane's hierarchy picture[6] of these states, Laughlin quasielectrons (QE's) fill (the same) fraction ν of their LL. The most striking conclusion from Pan's discovery is that the CF's (or QE's) can also form incompressible states when partially filling a LL. This could not be predicted by a simple analogy with known fractional electron liquids (Laughlin,[3] Jain,[4] or Moore-Read[7] states), because of a different form of QE-QE interaction,[8–10] therefore yielding qualitatively different QE-QE correlations.

Although several numerical studies of interacting QE's have been reported[10–13] and ideas such as CF flavor mixing,[14] QE pairing,[15,16] or stripes[17] were invoked, the correlations responsible for the FQHE at $\nu_e=\frac{4}{11}$ and $\frac{3}{8}$ are not yet understood. It has not even been settled if these FQH states are isotropic, and the energies of liquid and solid phases were compared recently[20] (although the Laughlin form was arbitrarily assumed for the liquid).

Sometimes overlooked is a general connection[18,19] between the form of Haldane pseudopotential,[21] the occurrence of Laughlin correlations, and the validity of the CF transformation. Actually, the form of QE-QE interaction is known from independent calculations,[8–10] and Laughlin correlations among the QE's have been ruled out using both a general pseudopotential argument[9] and a direct analysis of many-QE wave functions.[12] In this paper we refer to the following well-established facts.

(i) The QE-QE Haldane pseudopotential[21] is known from exact diagonalization of the Coulomb interaction among electrons in the lowest LL.[8–10] Since there are no unchecked assumptions in such a calculation, it must be regarded as a "numerical experiment." Neither finite-size errors, lowest-LL restriction, finite 2DEG width, nor other details of realistic experimental systems affect the dominant feature of the pseudopotential which is the lack of strong QE-QE repulsion at short range.

(ii) The QE's do not[9,12] have Laughlin correlations at $\nu=\frac{1}{3}$ corresponding to $\nu_e=\frac{4}{11}$. The Moore-Read half-filled state is not[12,22] an adequate description of QE-QE correlations at $\nu=\frac{1}{2}$ corresponding to $\nu_e=\frac{3}{8}$.

(iii) A sequence of nondegenerate finite-size QE ground states with a gap, extrapolating to $\nu=\frac{1}{3}$, has been found[12] on a sphere. Although spherical geometry is not adequate for studying crystal or other broken-symmetry phases, the identified states appear incompressible and have the lowest energy of all QE liquids (considerably below the Laughlin state).

To address the problem of correlations at $\nu_e=\frac{4}{11}$, $\frac{3}{8}$, and $\frac{5}{13}$ we calculate pair-distribution functions $g(r)$ in the incompressible liquid ground states of up to $N=14$ QE's. Their comparison with the (known) curves of the Laughlin and Moore-Read states implies a different nature of the QE correlations in these FQH states. It shows that their incompressibility cannot be explained by a simple analogy between the QE and electron liquids, and suggests that different wave functions need to be proposed for correlated CF's. Unfortunately, the calculated $g(r)$ are of little help in a precise definition of these wave functions, even though some qualitative statements can be made about the QE correlations.

From our finite-size results we identify and analyze the size-independent features in $g(r)$, the $\sim r^2$ behavior at short range and a shoulder at a medium range, and argue that they are consistent with the idea[12] of QE cluster formation. Short- and long-range contributions to $g(r)$ are found, describing correlations between the QE's from the same or different clusters. Both intra- and intercluster QE-QE correlations depend rather weakly on ν. The average size of the clusters is estimated; it seems that the QE's form pairs at $\nu=\frac{1}{3}$ and triplets at $\nu=\frac{1}{2}$. A similar analysis of $g(r)$ carried out for the Moore-Read state reveals a qualitatively different behavior.

II. MODEL

A. Haldane sphere

The numerical calculations have been carried out in Haldane's spherical geometry,[6] convenient for the exact study of short-range correlations. In this model, the lowest LL for particles of charge q is a degenerate shell of angular momentum $l=Q$. Here $2Q$ is the strength of the Dirac monopole in the center of the sphere defined in the units of elementary flux $\phi_0=hc/q$ as $2Q\phi_0=4\pi R^2 B$, the total flux of the magnetic field B through the surface of radius R. Using the usual definition of the magnetic length $\lambda=\sqrt{\hbar c/qB}$, this can be written as $l\lambda^2=R^2$. In the following, λ denotes the QE magnetic length corresponding to the fractional charge $q=-e/3$.

The relative ($\mathcal{R}$) and total (L) pair angular momenta are related via $L=2l-\mathcal{R}$. For fermions, $\mathcal{R}$ is an odd integer, and it increases with increasing average pair separation $\sqrt{\langle r^2\rangle}$. The interaction (within the lowest LL) is entirely determined by the Haldane pseudopotential defined as the pair-interaction energy V as a function of $\mathcal{R}$.

B. Exact diagonalization

Recently, we have identified[12] the series of finite-size spin-polarized states that in the thermodynamic limit describe the FQHE at $\nu_e=\frac{4}{11}$ and $\frac{3}{8}$. To do so, we have carried out extensive exact-diagonalization calculations for interacting QE's (particles in the second CF LL). On the Haldane sphere, N fermionic QE's were confined in a standard way to an angular momentum shell of degeneracy $\Gamma=2l+1$, corresponding to the QE filling factor $\nu\sim N/\Gamma$, and the Haldane QE-QE pseudopotential $V(\mathcal{R})$ was taken from earlier calculations.[8–10]

Regardless of the electron layer width w, magnetic field B, or other experimental parameters, the dominant feature of $V(\mathcal{R})$ is strong repulsion at $\mathcal{R}=3$. This feature alone determines the wave functions at $\frac{1}{3}\leqslant\nu\leqslant\frac{1}{2}$ (with the QE-QE correlations consisting of maximum possible avoidance of the Haldane pair amplitude $\mathcal{G}$ at $\mathcal{R}=3$), which are hence virtually insensitive to the (sample-dependent) details of $V(\mathcal{R})$. This justifies model calculations using the $V(\mathcal{R})$ of Refs. 8–10. Actually, a model pseudopotential as simple as $V=\delta_{\mathcal{R},3}$ is sufficient to reproduce correct correlations and incompressibility at $\nu_e=\frac{4}{11}$ or $\frac{3}{8}$.

III. NUMERICAL RESULTS

A. Energy spectra

The numerical results carried out for $N\leqslant 14$ (two sample spectra are displayed in Fig. 1) showed[12] a sequence of nondegenerate (i.e., at the total angular momentum $L=0$) ground states at $2l=N/\nu-\gamma$ with $\nu=\frac{1}{3}$ and $\gamma=7$. The significant and well-behaved (as a function of N) excitation gap along this sequence strongly suggests that it represents the infinite $\nu_e=\frac{4}{11}$ FQH state observed in experiment.[1] The value $\gamma\neq 3$ precludes Laughlin correlations among QE's in this state (earlier ruled out indirectly, based on the form of QE-QE pseudopotential[9]), i.e., the idea that the $\nu_e=\frac{4}{11}$ state is simply a Haldane hierarchy state of Laughlin-correlated CF's. While the exact correlations in this (known only numerically for a few consecutive N) ground state have not yet been defined, their vanishing degeneracy ($L=0$) implies that they describe a QE liquid, rather than a broken-symmetry state (such as liquid-crystal nematic states proposed[23] in the context of the FQHE at different values of ν).

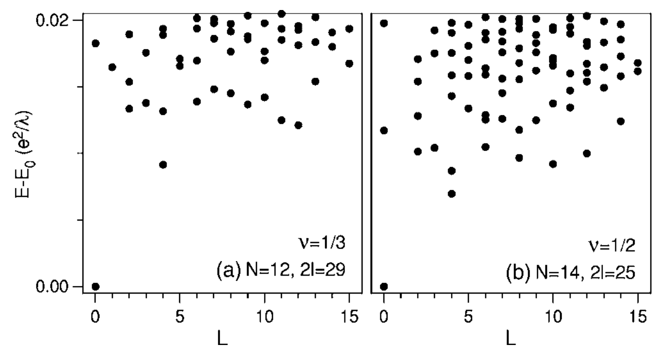

FIG. 1. Excitation energy spectra (energy E as a function of total angular momentum L; E_0 is the ground-state energy) of N interacting QE's on a sphere, at the values of CF LL degeneracy $\Gamma=2l+1$ corresponding to the incompressible ground states at the QE filling factors $\nu=1/3$ (a) and $1/2$ (b).

Another sequence was anticipated at $2l=2N-\gamma$ to represent the infinite $\nu_e=\frac{3}{8}$ FQH state. However, the only ground state with a significant gap and remaining outside of the $\nu=\frac{1}{3}$ sequence (or its particle-hole symmetric $\nu=\frac{2}{3}$ sequence at $2l=\frac{3}{2}N+2$) occurs[12] for $N=14$ and $2l=25$ (and it also has $L=0$). These values of $(N,2l)$ happen to belong to a $2l=2N-3$ series representing the Moore-Read (Pfaffian) paired state, but the overlap between the two turns out nearly zero.[12,22] Moreover, the ground states for the two neighboring even (as appropriate for a hypothetically paired state) values of $N=12$ and 16 (and $2l=21$ and 29) have $L>0$ and no gap, the value of $2l=17$ for $N=10$ coincides with the $\nu=\frac{2}{3}$ sequence (so that only for $N>8$ can the filling factor ν be meaningfully assigned), and we are unable to compute the spectra for $N\geqslant 18$. Nevertheless, despite little evidence available from numerical diagonalization, the ground state for $N=14$ and $2l=25$ (and its particle-hole counterpart at $N=12$ and the same $2l=25$) may possibly represent the $\nu_e=\frac{3}{8}$ FQH state (i.e., have similar correlations causing incompressibility).

B. Pair-distribution functions

The QE-QE pair-distribution functions $g(r)$ have been calculated for the incompressible many-QE ground states as expectation values of the appropriate pair interaction,

$$g(r)=(2/N)^2\langle\delta(R\theta-r)\rangle. \quad (1)$$

Here, θ is the relative angle on a sphere, so that r measures interparticle distance along the surface (rather than chord distance). More accurately, r is the distance between the centers of extended QE's (note that in the calculation of many-QE wave functions, the system of QE's is mapped onto the lowest LL of point charges interacting through an effective pseudopotential). The prefactor in Eq. (1) ensures proper normalization, $g(\infty)\to 1$. Denoting the infinitesimal area by $dS=2\pi R^2 d(\cos\theta)$ or (in magnetic units) by $ds=dS/2\pi\lambda^2$, we get an equivalent normalization condition

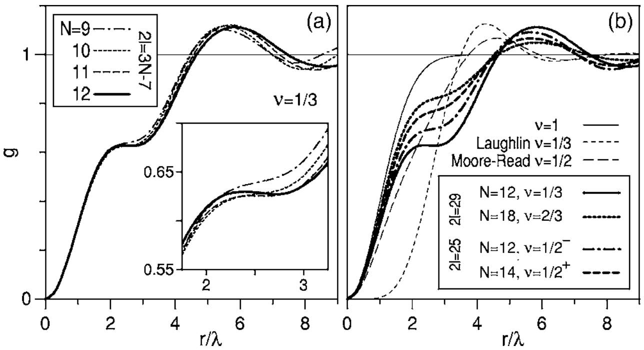

FIG. 2. QE-QE pair-distribution functions $g(r)$ of the incompressible ground states at different QE filling factors ν. (a) Curves for $\nu=1/3$ and different QE numbers N; (b) curves for QE's at different ν (thick lines) compared to some known incompressible states of electrons.

$$\int [1-g(r)]ds = \frac{2l}{N} \to \nu^{-1} \tag{2}$$

in large systems. Since $ds = l\, d(\cos\theta)$, a "local filling factor" can also be defined as $\nu(r) = dN/ds = (N/2l)g(r)$, and it satisfies $\nu(\infty)=\nu$ and $\int \nu(r)ds = N-1$.

The results for the $\nu=\frac{1}{3}$ sequence at $2l=3N-7$ are shown in Fig. 2(a). Similarity of all four curves is evident, indicating a size-independent form of correlations (hence, describing an infinite system), with a well-developed shoulder around $r\approx 2.5\lambda$. Similar shoulders occur in $g(r)$ of all incompressible ground states at $\nu=\frac{2}{3}$ or $\frac{1}{2}$ (the $\nu=\frac{2}{3}$ sequence at $2l=\frac{3}{2}N+2$ is obtained from $2l=3N-7$ by replacing N with $\Gamma-N$, while at $\nu=\frac{1}{2}$ there are two particle-hole conjugate sequences at $2l=2N-3$ and $2N+1$, denoted by $\nu=\frac{1}{2}^{\pm}$). The four curves representative of $\nu=\frac{1}{3}$, $\frac{2}{3}$, and $\frac{1}{2}^{\pm}$ are shown in Fig. 2(b). They are all clearly different from those marked with thin lines and describing correlations known for other incompressible FQH states (full LL, Laughlin $\nu=\frac{1}{3}$ state, or Moore-Read half-filled state). This is a direct indication of the different nature of QE-QE correlations responsible for the FQHE at $\nu_e=\frac{4}{11}$ and $\frac{3}{8}$.

Let us stress that although the QE-QE interactions are not known with great accuracy, the correlation functions in Fig. 2 are rather insensitive to the details of $V(\mathcal{R})$, as long as the dominant repulsion occurs at $\mathcal{R}=3$ (which seems to be universally true in the systems studied experimentally). This insensitivity is reminiscent of the Laughlin wave function, which also very accurately describes the actual $\nu=\frac{1}{3}$ ground state for a wide class of electron-electron pseudopotentials. However, while the avoidance of $\mathcal{R}=1$ by the electrons in the lowest LL can be elegantly described by flux attachment in the CF picture, no similar model has been proposed yet for the avoidance of $\mathcal{R}=3$ by the QE's. Therefore, knowing the $g(r)$ curves of QE's and understanding their correlations, we still cannot write their wave functions.

C. Gaussian deconvolution

The curves of Fig. 2(b) can be accurately deconvoluted using Gaussians, $G(r/\lambda)=A\exp[-(r/\lambda-\delta)^2/2\sigma^2]$. This is shown in Fig. 3 where the symbols mark the exact data of Fig. 2(b) and the lines give the (nearly perfect) fits using three Gaussians, $g=1-G_0-G_1-G_2$ (sufficient for $r\leqslant 6\lambda$). The fitted values of $[A_i,\delta_i,\sigma_i]$ for all four curves are listed in Table I. Note that $A_0=1$, $\delta_0=0$, and $\delta_1=3$ for all curves (the last value being least obvious, but probably resulting from the avoidance of the same $\mathcal{R}_3=3$ by the QE's at all values of ν). The values of the G_2 parameters are not very meaningful when the next term in the approximation (G_3) is neglected. The clearest difference between the four curves is in A_1.

D. Short- and long-range deconvolution

It appears more physically meaningful to decompose $g(r)$ into $g_0=1-\exp(-r^2/2\lambda^2)$, describing a full lowest LL,[24] and a (properly normalized) "remainder" $g_{\rm diff}$,

$$g(r) = \alpha\, g_0(r) + (1-\alpha)g_{\rm diff}(r). \tag{3}$$

For each $g(r)$, the parameter α is calculated as the limit of g/g_0 at $r\to 0$. It is clear from Fig. 4(a) that $g(r)$ is accurately approximated by $\alpha\, g_0(r)$ within a finite area or a radius $\sim\lambda$ for all four ground states of Fig. 2(b). The numerical values of α are 0.772, 0.804, 0.856, and 0.899 for $\nu=\frac{1}{3}$, $\frac{1}{2}^-$, $\frac{1}{2}^+$, and $\frac{2}{3}$, respectively. Evidently, α is size dependent (e.g., the pair of values for $\nu=\frac{1}{2}^{\pm}$ must converge to the same thermodynamic limit).

The four curves $g_{\rm diff}(r)$ calculated from Eq. (3) are plotted in Fig. 4(b). Symbols and lines mark the exact data and the three-Gaussian fits of Table I, respectively. We note the following. (i) For the pairs of particle-hole conjugate states ($N=12,18$ at $2l=29$ and $N=12,14$ at $2l=25$), the $g_{\rm diff}(r)$ curves are *identical*. (ii) The curves obtained for $\nu=\frac{1}{3}$ and $\frac{1}{2}$ are very similar (and possibly identical in large systems); they all vanish at short range and have a minimum at $r\approx 3\lambda$ and a maximum at $r\approx 5.5\lambda$.

TABLE I. Gaussian deconvolution parameters for QE-QE pair-distribution functions shown in Figs. 2(b) and 3.

ν	A_0	δ_0	σ_0	A_1	δ_1	σ_1	A_2	δ_2	σ_2
1/3	1	0	1.0989	0.3450	3	0.9412	−0.1199	5.6905	1.0298
2/3	1	0	1.0419	0.1535	3	0.9361	−0.0530	5.6655	0.9987
$1/2^+$	1	0	1.0626	0.2034	3	0.9475	−0.0741	5.4041	1.1011
$1/2^-$	1	0	1.0896	0.2755	3	0.9431	−0.1005	5.4156	1.0903

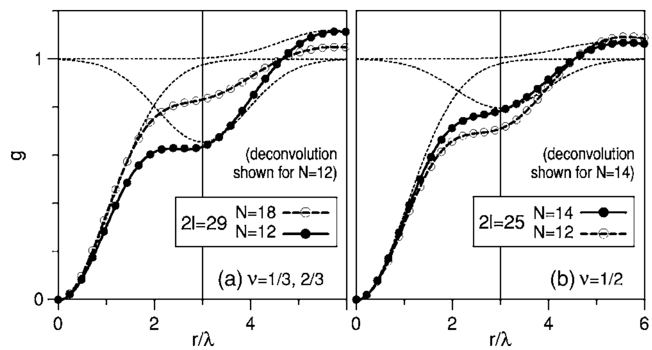

FIG. 3. Gaussian deconvolution of the QE-QE pair distribution functions $g(r)$: dots, data of Fig. 2(b); lines, fits.

IV. DISCUSSION

A. QE clustering

Some information about the form of QE-QE correlations can be easily deduced from the form of interaction pseudopotential $V(\mathcal{R})$, which is simply the interaction Hamiltonian defined only for those pair states allowed in the lowest LL. In low-energy many-body states the particles generally tend to avoid pair eigenstates with high interaction energy, which means minimization of the corresponding Haldane pair amplitude $\mathcal{G}$. If the repulsion V decreases sufficiently quickly[18] as a function of $\mathcal{R}$ (the exact criterion being[19] that V decreases sublinearly as a function of $\sqrt{\langle r^2\rangle}$), the smallest value of $\mathcal{R}=1$ is avoided. This Laughlin type of correlation is elegantly described by attachment of $2p=2$ fluxes to each particle in the CF transformation. In a Laughlin-correlated state, each particle avoids being close to any other particle (as much as possible at a given finite ν).

When short-range repulsion weakens (V at $\mathcal{R}=1$ decreases compared to V at $\mathcal{R}\geqslant 3$), Laughlin correlations disappear and can be replaced by pairing or formation of larger clusters. Pairs[15,16] or clustering[12] were suggested by several authors for the QE's. This idea was justified by the observation that the QE-QE pseudopotential nearly vanishes at $\mathcal{R}=1$ and is strongly repulsive at $\mathcal{R}=3$, causing an increase of $\mathcal{G}(1)$ and a simultaneous decrease of $\mathcal{G}(3)$ compared to the Laughlin-correlated state.[12]

The assumption that QE's form clusters naturally explains the shoulder in $g(r)$, and allows one to interpret g_0 and g_{diff} as the intra- and intercluster QE-QE correlations, i.e. the short- and long-range contributions to g, corresponding to the QE pairs belonging to the same or different clusters, respectively. The vanishing of $g_{\mathrm{diff}}(r)$ at short range reflects isolation of QE's belonging to different clusters. The reason why g_{diff} is not positive definite is that intracluster correlations are accurately described by g_0 only within a certain radius. In other words, the actual intercluster contribution to g is not *exactly* given by g_{diff} defined by Eq. (3). Nevertheless, the following two conclusions remain valid: (i) the intra- and intercluster QE-QE correlations are similar at $\nu=\frac{1}{3}$, $\frac{1}{2}$, and $\frac{2}{3}$, with the respective correlation-hole radii $\varrho_0\sim\lambda$ and $\varrho_1\sim 4\lambda$; and (ii) the cluster size K depends on ν.

A similar form of $g(r)$ was found[23] for broken-symmetry Laughlin states, in which the shoulder results from angular averaging of an anisotropic function $g(r,\phi)\sim r^2$ or r^6, depending on ϕ. However, the present case of QE's is different, because $g(r)$ is isotropic (wave functions have $L=0$) and the shoulders result from *radial* averaging of inter- and intracluster correlations (beginning as $\sim r^2$ and a higher power of r at short range, respectively).

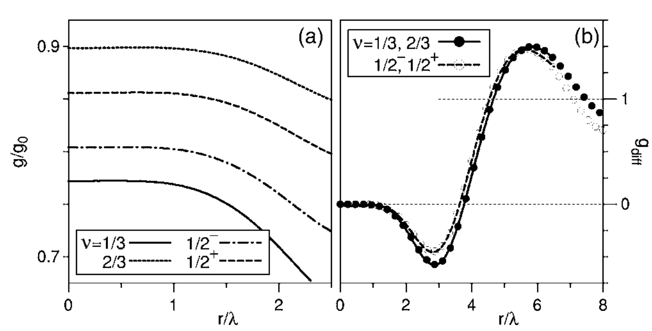

FIG. 4. (a) Ratio of QE-QE pair-distribution functions $g(r)$ to $g_0(r)$ of a full lowest LL for different incompressible QE ground states; (b) the "remainder" $g_{\mathrm{diff}}(r)$ defined by Eq. (3).

TABLE II. Parameters β_K of the short-range approximation $\nu(r)\sim\beta\, g_0(r)$ obtained for independent clusters of size K.

$2l$	β_2	β_3	β_4	β_5	β_6
25	0.2768	0.4196	0.5110	0.5765	0.6269
29	0.2730	0.4134	0.5029	0.5669	0.6159
60	0.2609	0.3938	0.4778	0.5372	0.5821
∞	0.2500	0.3763	0.4555	0.5110	0.5527

B. Average cluster size

In a clustered state, the (average) cluster size K is connected to α, and the form of g_{diff} depends on correlations between the clusters. The values of K at $\nu=\frac{1}{3}$ or $\frac{1}{2}$ can be estimated by comparison of the actual parameters α with those predicted for the hypothetical states of N particles arranged into N/K *independent* K-clusters. By independence of the clusters we mean that intercluster correlations do not affect the local filling factor $\nu(r)$ at short range. For a single cluster, which on a sphere is the K-particle state with the maximum total angular momentum $L=Kl-\frac{1}{2}K(K-1)$, the $\nu_K(r)$ depends on the surface curvature and thus (through $R/\lambda=\sqrt{l}$) on $2l$.

We have calculated the prefactors β_K of the short-range approximation $\nu_K(r)\approx\beta_K g_0(r)$ for different values of K and $2l$ and listed some in Table II [note that $\nu_2(r)$ is known exactly]. These coefficients are to be compared with $\beta=(N/2l)\alpha$ of the incompressible N-QE states obtained from diagonalization. Of course, this approach is somewhat questionable as one generally cannot deduce the precise cluster size from the short-range behavior of $g(r)$ for the following reasons: (i) K is not a well-defined (conserved) quantum number; (ii) $\nu=\frac{1}{3}$ states occur for all N (not only those divisible by 2 or 3) which means that all clusters cannot have the same K; (iii) the parameters α and β are size dependent and

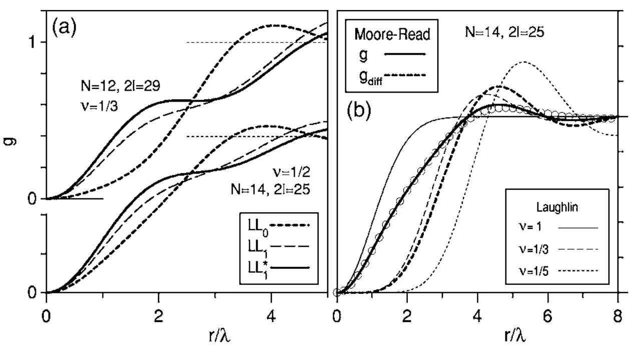

FIG. 5. (a) Pair-distribution functions $g(r)$ of lowest $L=0$ states of finite systems corresponding to $\nu=1/3$ and $1/2$, for pseudopotentials of electrons in the first and second LL, and of CF's in the second LL. (b) The total $g(r)$ and "remainder" $g_{\rm diff}(r)$ curves of the Moore-Read $\nu=1/2$ state; circles mark a fitting linear combination of the curves for Laughlin states.

their extrapolation to large systems is not very reliable based on the limited number of N-QE systems we are able to diagonalize; (iv) intercluster exchange of QE's makes the "independent-cluster" picture only an approximation.

Fortunately, we can use the Moore-Read states (known to be paired[7,22]) as a test. Our calculation (for details see Sec. IV C) for $N=14$ and $2l=25$ gives $\beta_{\rm MR}=0.336$, somewhat larger than β_2. Hence, we shall assume that β_K in general underestimates the actual value of β in a many-body K-clustered state.

For the QE's, we got $\beta=0.319\approx\beta_{\rm MR}$ for $N=12$ and $2l=29$ $(\nu=\frac{1}{3})$, and $\beta=0.479$ for $N=14$ and $2l=25$ ($\nu=\frac{1}{2}^+$; directly comparable with the Moore-Read state). With appropriate reservation, we can hence risk a hypothesis that QE's (on the average) form pairs at $\nu=\frac{1}{3}$ and triplets at $\nu=\frac{1}{2}$ (possible triplet formation might turn out especially intriguing in the context of parafermion statistics[25]).

C. Comparison with Moore-Read state

The evolution of $g(r)$ when going from the lowest electron LL to the second CF LL (i.e., from LL_0 to CF-LL_1) is clear when using a model pseudopotential $V_\zeta(\mathcal{R})=\zeta\,\delta_{\mathcal{R},1}+(1-\zeta)\delta_{\mathcal{R},3}$. For $\zeta\approx0$ or 1, the correlations (avoidance of $\mathcal{R}=1$ or 3) are insensitive to ζ, and both Laughlin and QE-QE correlations are accurately reproduced by V_0 and V_1, respectively. Modeling correlations among electrons in LL_1 (the second LL) is more difficult, because they are very sensitive to the exact form of $V(\mathcal{R})$ at the corresponding $\zeta\sim\frac{1}{2}$. As a result, the N-electron Coulomb eigenstates in LL_1 are more susceptible to finite-size errors than in LL_0 or CF-LL_1. In large systems, a good trial state is only known at $\nu=\frac{1}{2}$ (Moore-Read state), and much less is established about the correlations at $\nu=\frac{1}{3}$. Still, the $g(r)$ curves for electrons in LL_1 must certainly fall between the two extreme curves for $\zeta=0$ and 1 (and differ from both of them). This is shown in Fig. 5(a) for both $\nu=\frac{1}{3}$ and $\frac{1}{2}$.

The exact Moore-Read wave functions were calculated on a sphere for $N\leq14$ and $2l=2N-3=25$ by diagonalizing a short-range three-body repulsion.[22] In Fig. 5(a) we only plotted $g(r)$ for $N=14$ because the $N=12$ curve is too close to be easily distinguished. The values of $\alpha=0.602$ and 0.600 for $N=12$ and 14. The $g_{\rm diff}(r)$, also shown, is positive definite, very different from the QE curves in Fig. 4(b), and rather close to $g_1(r)$, where g_p describes a Laughlin $\nu=(2p+1)^{-1}$ state. Assuming $\alpha_{\rm MR}=\frac{3}{5}$ and expanding $g_{\rm diff}$ into g_1 and g_2 in accordance with Eq. (2) one obtains an approximate formula

$$g_{\rm MR}(r)\approx\frac{3}{5}g_0(r)+\frac{3}{10}g_1(r)+\frac{1}{10}g_2(r),\qquad(4)$$

marked with the circles in Fig. 4(b), that appears to be quite accurate (the largest finite-size error is in g_2 calculated for only $N=8$, while g_1 is for $N=12$).

The fact that $g_{\rm diff}$ is positive and rather featureless (similar to g_p) for the Moore-Read wave function is in contrast with the result for QE's. This difference may indicate that the QE clusters cannot be understood literally as Moore-Read pairs. Indeed, even the lack of correlation between the occurrence of $L=0$ ground states (or size of the excitation gap) and the divisibility of N by $K=2$ or 3 precludes such a simple picture. The fact that $g_{\rm diff}(r\sim3\lambda)<0$ could mean that the average relative (with respect to center of mass) angular momentum $\mathcal{R}_K$ of the QE clusters is much larger than $\mathcal{R}_K^{\rm min}=\frac{1}{2}K(K-1)$. Certainly, $\mathcal{R}_K$ is only conserved for an isolated cluster, but it is possible that the QE clusters are more relaxed due to cluster-cluster interaction than the Moore-Read pairs are. This would make g_0 underestimate the radius of the actual intracluster QE-QE correlation hole, and explain the negative sign of $g_{\rm diff}$.

V. CONCLUSION

From exact numerical diagonalization on Haldane sphere, we obtained the energy spectra and wave functions of up to $N=14$ interacting Laughlin QE's (CF's in the second LL). We identified the series of finite-size liquid ground states with a gap, which extrapolate to the experimentally observed incompressible FQH states at $\nu_e=\frac{4}{11}$, $\frac{3}{8}$, and $\frac{5}{13}$. In these states, we calculated QE-QE pair-distribution functions $g(r)$, and showed that they increase as $\sim r^2$ at short range and have a pronounced shoulder at a medium range. This behavior supports the idea of QE cluster formation, suggested earlier from the analysis of the QE-QE interaction pseudopotential. The $g(r)$ is decomposed into short- and long-range contributions, interpreted as correlations between the QE's from the same or different clusters. The intracluster contribution to $g(r)$ is that of a full LL, and the remaining term identified with the intercluster QE-QE correlations appears to be the same in all three $\nu=\frac{1}{3}$, $\frac{1}{2}$, and $\frac{2}{3}$ states. The (average) cluster size on the other hand does depend on ν, and we present arguments which suggest that the QE's form pairs at $\nu=\frac{1}{3}$ and triplets at $\nu=\frac{1}{2}$.

The qualitative difference between the $g(r)$ curves obtained here for correlated CF's and those known for the Laughlin and Moore-Read liquids of electrons is another indication that the origin of incompressibility at $\nu_e=\frac{4}{11}$, $\frac{3}{8}$, and $\frac{5}{13}$ is different. Of other hypotheses invoked in literature and mentioned here in the Introduction, the broken-symmetry states cannot be excluded by our calculation in spherical geometry. However, we anticipate that the QE's form a liquid (studied in this paper) also in experimental samples, because of the whole series of isotropic ground states with a gap occurring in finite systems of different size.

ACKNOWLEDGMENTS

The authors thank W. Pan, W. Bardyszewski, and L. Bryja for helpful discussions. This work was supported by Grant No. DE-FG 02-97ER45657 of the Materials Science Program, Basic Energy Sciences of the U.S. Dept. of Energy, and Grant No. 2P03B02424 of the Polish KBN.

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