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Numerical evidence for non-Abelian quantum liquids in the lowest Landau level

Arkadiusz Wójs
Institute of Physics, Wrocław University of Technology, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland
and TCM Group, Cavendish Laboratory, University of Cambridge, Cambridge CB3 0HE, United Kingdom
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Signatures of non-Abelian statistics are sought in correlated liquids of composite fermions (CFs) responsible for a "second generation" of fractional quantum Hall effect. The hierarchy stems from the zero-energy state of a model CF interaction by means of a flux attachment procedure converting this uniquely correlated state of "first-generation" CFs to a filled shell of second-generation CF*'s. Quasiholes of this state do not obey Abelian statistics. The hierarchy is confirmed numerically, including known states at filling factors $\nu_e=4/11$ and 3/8 and a hypothetical state at $\nu_e=9/25$.

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The theory of Laughlin[1] for fractional quantum Hall (FQH) effect[2] invokes the concept of incompressible quantum liquids (IQLs) formed by two-dimensional (2D) electrons in the lowest Landau level (LL). The quasiparticle (QP) excitations of the IQL carry fractional electric charge (e.g., $q=\pm\frac{1}{3}e$ at LL filling factor $\nu_e=\frac{1}{3}$). It was further realized[3] that Laughlin QPs are neither fermions nor bosons but "anyons" characterized by complex exchange phase $e^{i\theta}$ and allowed exclusively in two dimensions.[4] However, Chern-Simons transformation (attachment of magnetic flux) enables transmutation of statistics in two dimensions.[5] This justifies fermionic description of Laughlin QPs in the "composite fermion" (CF) theory of Jain[6] for the FQH effect.

Even more unusual statistics was later identified for quasihole (QH) excitations of the Moore-Read state,[7] arguably responsible for FQH effect in a half-filled second LL.[8] Since the Hilbert space of Moore-Read QHs located at fixed positions is degenerate, their adiabatic exchange is represented by a matrix (instead of merely a phase factor). Since matrix exchange operations do not generally commute, such statistics is called "non-Abelian."

Interest in non-Abelian states was revived by the idea of fault-tolerant "topological quantum computation."[9] However, the focus has been almost exclusively on the Moore-Read model state,[10] whose relevance for experimental FQH systems at $\nu_e=\frac{5}{2}$ is still debated.[11,12]

In this Rapid Communication we postulate a *non-Abelian electron liquid in the lowest LL*. The hierarchy it generates through a flux attachment transformation includes confirmed[13] FQH states at $\nu_e=\frac{4}{11}$ and $\frac{3}{8}$, which in the model of Jain[6] correspond to a partially filled second LL of CFs.

We consider N fermions of charge q confined to a Haldane sphere of unit radius.[14] The magnetic monopole of strength $2Q$ (i.e., flux $4\pi B=2Q\phi_q$, where $\phi_q=hc/q$ is the flux quantum) produces isotropic radial field B yielding magnetic length scale $\lambda\equiv\sqrt{\hbar c/qB}=Q^{-1/2}$ at the surface. The sth LL (called LL_s) is a multiplet of single-particle angular momentum $\ell=Q+s$. Interaction Hamiltonian in an isolated LL is determined by Haldane pseudopotential $V(\mathcal{R})$, defined[15] as dependence of pair interaction energy on relative pair angular momentum (for K particles: $\mathcal{R}\equiv K\ell-L$, where L is total angular momentum; for two identical fermions: $\mathcal{R}=1,3,\ldots$).

The Haldane sphere is a useful model for extended many-body problems in a partially filled LL, assuming that correlations are isotropic and have relatively short length. This is usually true of FQH liquids. For example, correlations of the Laughlin $\nu_e=\frac{1}{3}$ state are induced by the first coefficient of the Coulomb pseudopotential in LL_0. They mean avoidance of the pair state at the minimum $\mathcal{R}=1$ or, more formally, vanishing of Haldane pair amplitude $\mathcal{G}$ (Ref. 15) at $\mathcal{R}=1$. In terms of wave functions, each electron binds two additional vortices, as described by the Laughlin-Jastrow prefactor $\Pi_{i<j}(z_i-z_j)^2$ (where z's are complex coordinates). This is elegantly captured by the theory of Jain[6] in which correlated electrons convert into nearly free CFs by binding some of the external magnetic field B in the form of flux tubes. Flux $2\phi_e$ pointing opposite to B is attached to each electron, leaving a reduced effective field $B_{\mathrm{CF}}=B-2\varrho\phi_e$ (ϱ being 2D concentration) seen by the CFs and corresponding to an increased effective filling factor $\nu_{\mathrm{CF}}=(\nu_e^{-1}-2)^{-1}$.

The "second generation" of FQH states[13] corresponds to a partially filled CF-LL_1 (second LL of CFs). The strongest states $\nu_e=\frac{4}{11}$ and $\frac{3}{8}$ have $\nu_{\mathrm{CF}}=\frac{4}{3}$ and $\frac{3}{2}$, i.e., $\nu=\frac{1}{3}$ and $\frac{1}{2}$ partial fillings of CF-LL_1. In contrast to "first-generation" Laughlin/Jain states, their incompressibility depends on "residual" CF interaction. Effective pseudopotential in CF-LL_1 is dominated by repulsion at $\mathcal{R}=3$,[16,17] making CF correlations at

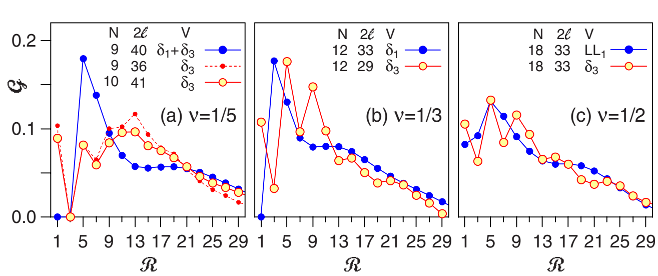

FIG. 1. (Color online) (a) Discrete correlation functions (Haldane amplitude $\mathcal{G}$ versus relative pair angular momentum $\mathcal{R}$) for $\nu=\frac{1}{5}$ nondegenerate zero-energy ground states of model interaction $V=\delta_3$ for $N=9$ and 10 particles on a Haldane sphere ($2\ell=5N-9$) compared to Laughlin state ($V=\delta_1+\delta_3$, $2\ell=5N-5$). (b) and (c) Analogous plots for $\nu=\frac{1}{3}$ and $\frac{1}{2}$: nondegenerate ground states of $V=\delta_3$ at $2\ell=3N-7$ and $2N-3$ compared to Laughlin state ($V=\delta_1$ and $2\ell=3N-3$) and nondegenerate Coulomb ground state in LL_1, respectively.

TABLE I. Total angular momentum (L) multiplets with zero amplitude $\mathcal{G}(3)$ at the relative pair state $\mathcal{R}=3$ [i.e., the exact zero-energy states of the model pseudopotential $V=\delta_3(\mathcal{R})$] for $N=8$ particles with different shell angular momenta ℓ.

2ℓ	L
32	
33	$0^2\ 2^2\ 4\ 6\ 8$
34	$0^3\ 1\ 2^4\ 3^3\ 4^5\ 5^2\ 6^4\ 7^2\ 8^2\ 9\ 10\ 12$
35	$0^8\ 1^2\ 2^{13}\ 3^8\ 4^{18}\ 5^{10}\ 6^{17}\ 7^9\ 8^{13}\ 9^6\ 10^8\ 11^3\ 12^4\ 13\ 14^2\ 16$

$\nu=\frac{1}{3}$ or $\frac{1}{2}$ distinct from electron correlations at the same filling ν_e of LL_0 or LL_1. The tendency of CFs to minimize the pair amplitude $\mathcal{G}(3)$ instead of $\mathcal{G}(1)$ was demonstrated (by direct calculation of pair and triplet amplitudes)[18] to be equivalent to a form of CF pairing.

For description of correlated CFs, an intuitive model analogous to the CF description by Jain[6] of Laughlin-correlated electrons would be useful. Hence, we seek conversion of an incompressible many-CF state at $\nu=\frac{1}{3}$ or $\frac{1}{2}$ to a filled shell of (essentially) noninteracting hypothetical fermions to be called "second-generation CFs" or, shortly, CF*'s.

First, we will identify the maximum-density state with $\mathcal{G}(3)=0$, i.e., nondegenerate zero-energy ($E=0$) ground state of a model pseudopotential $V(\mathcal{R})=\delta_3(\mathcal{R})\equiv\delta_{\mathcal{R},3}$. From exact diagonalization of $N\leq 10$ fermions interacting through $V=\delta_3$ in LLs with different ℓ we find such $E=L=0$ series at $2\ell=5N-9$, extrapolating to $N/2\ell\to\nu=\frac{1}{5}$ in large systems. CF pairing in this state is evident from the amplitudes $\mathcal{G}(\mathcal{R})$ plotted in Fig. 1. Moreover, comparison of correlation energies shows that it is favored over the Laughlin $\nu=\frac{1}{5}$ state of the CFs.

Conversion from $2\ell=5N-9$ to $2\ell^*=N-1$ of a filled CF* shell is achieved by the following transformation:

$$2\ell^* = 2\ell - 4(N-2). \tag{1}$$

Attributing degeneracy of CF-LL_1 to fictitious magnetic flux, Eq. (1) can be interpreted as attachment of $p=4$ flux quanta to each CF. Furthermore, the factor $(N-2)$ suggests that each CF* sees an average flux from all but one other CF, which simply reflects the CF pairing.

Notably, the $2\ell=5N-9$ series of ground states includes both even and odd CF numbers, undermining the "CF pairing" interpretation of Eq. (1). However, no particular CF correlation was assumed in the formulation of Eq. (1), which instead followed directly from the analysis of "numerical experiments" (spectra of $V=\delta_3$).

TABLE II. Angular momentum multiplets of $K\leq 4$ Abelian quasiholes in a shell of angular momentum ℓ^* (to be compared with data from Table I for $N=8$ particles and different ℓ's).

2ℓ	K	$2\ell^*$	L
32	1	8	4
33	2	9	$0\ 2\ 4\ 6\ 8$
34	3	10	$0\ 2\ 3\ 4^2\ 5\ 6^2\ 7\ 8\ 9\ 10\ 12$
35	4	11	$0^2\ 2^3\ 3\ 4^4\ 5^2\ 6^4\ 7^2\ 8^4\ 9^2\ 10^3\ 11\ 12^2\ 13\ 14\ 16$

Let us turn to elementary excitations. QHs of a nondegenerate $E=0$ ground state can be identified as degenerate $E=0$ eigenstates in a system with added flux.[19] We computed the spectra of $V=\delta_3$ for $N\leq 10$ CFs at $2\ell=5N-9+K$ with $K\leq 4$. In Table I we list all $E=0$ multiplets L^η for $N=8$ (η counts the number of multiplets at each L). For comparison, in Table II we show the multiplets predicted from CF transformation (1), i.e., for K fermion QHs in a shell with $\ell^*=\frac{1}{2}(N-1+K)$.

Surprisingly, Table I shows no states at $2\ell=32$ in contrast to prediction of a single QH ($K=1$) in Table II. Evidently, addition of a single flux quantum to the nondegenerate $E=0$ ground state at $2\ell=5N-9$ does not produce a degenerate $E=0$ band that might be interpreted as LL degeneracy of a QH. This is clearly different from other known (Laughlin, Jain, or Moore-Read) IQLs.

At $2\ell>32$, Table I shows a growing number of $E=0$ states, in each case containing all multiplets from Table II. The maximum total angular momentum of K QHs, $\Lambda=K\ell^*-\frac{1}{2}K(K-1)=\frac{1}{2}KN$, always correctly predicts the maximum L in Table I, supporting the CF picture of QH excitations. Fermi statistics of QHs is conventional since the same sets of L multiplets could result for K bosons with angular momentum $\ell_B^*=\ell^*-\frac{1}{2}(K-1)=\frac{1}{2}N$. However, the occurrence of additional $E=0$ states beyond those predicted for K fermions/bosons implies that the QHs considered here cannot be described as Abelian (regardless of the choice of K and ℓ^*). This argument was raised earlier[19] for non-Abelian QHs of other known paired IQLs. In those states, knowledge of the many-body wave function allowed expression of total dimensions of the Hilbert spaces, $\mathcal{D}=\Sigma\,\eta(2L+1)$, by direct account for (non-Abelian) exchanges of the QHs. Here, the values of $\mathcal{D}$ are different, and (not knowing an explicit formula) we list them in Table III along with $\mathcal{D}_a=\binom{N+K}{N}$ for K Abelian QHs. For Moore-Read state, $\mathcal{D}/\mathcal{D}_a\to 2^{K/2-1}$ in large

TABLE III. Dimensions $\mathcal{D}$ of zero-energy subspaces for $N\leq 9$ fermions with interaction $V=\delta_3(\mathcal{R})$, compared with dimensions $\mathcal{D}_a$ of the corresponding spaces of K Abelian quasiholes.

K	2				3				4			
N	6	7	8	9	6	7	8	9	6	7	8	9
$\mathcal{D}$	29	39	51	65	105	182	295	452	376	790	1553	2878
$\mathcal{D}_a$	28	36	45	55	84	120	165	220	210	330	495	715

TABLE IV. Numbers η of zero-energy angular momentum multiplets L in the spectrum of model interaction $V=\delta_3(\mathcal{R})$ for different particle numbers $N\leq 9$ and shell angular momenta $\ell=\frac{1}{2}(5N-9+K)$; $\Lambda=\frac{1}{2}KN$ (see text).

		η for $\Lambda-L$																		
K	N	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	6	**1**		**1**		1		**2**												
	7	**1**		**1**		1		**2**												
	8	**1**		**1**		1		**2**		**2**										
	9	**1**		**1**		1		**2**		**2**										
3	6	**1**		**1**	**1**	**2**	1	3		2										
	7	**1**		**1**	**1**	**2**	**2**	3	2	3	2	1								
	8	**1**		**1**	**1**	**2**	**2**	**4**	2	5	3	4	1	3						
	9	**1**		**1**	**1**	**2**	**2**	**4**	**3**	5	5	5	4	5	1					
4	6	**1**		**2**	**1**	**4**	2	6	2	6	2	4		4						
	7	**1**		**2**	**1**	**4**	**3**	7	5	10	5	10	5	7	1	5				
	8	**1**		**2**	**1**	**4**	**3**	**8**	6	13	9	17	10	18	8	13	2	8		
	9	**1**		**2**	**1**	**4**	**3**	**8**	**7**	14	12	21	17	28	18	28	16	20	5	11

systems.[19] Here, this ratio is higher, although finite-size data in Table III are insufficient for extrapolation.

Dependence of multiplicities η on system size appears simpler when they are considered as a function of $\mathcal{R}^*\equiv K\ell^*-L$ (relative angular momentum of K QHs) instead of L. Actually, to make Table IV more compact we chose $\Lambda-L=\mathcal{R}^*-\frac{1}{2}K(K-1)$ instead of $\mathcal{R}^*$ itself. Clearly, multiplicities η become size independent at larger N, and the $N=\infty$ limits $\eta(K,\mathcal{R}^*)$ have already been reached for $N\leq 9$ at those few smallest $\mathcal{R}^*$'s marked with boldface.

Comparison with multiplicities $\eta_a(K,\mathcal{R}^*)$ predicted for K Abelian QHs is made in Table V. The differences $\eta-\eta_a$ were calculated for $N=9$, but for the shown small values of $\Lambda-L$ they correctly describe an infinite (planar) system. Similar to state counts for fixed QH positions,[19] $\eta-\eta_a$ gives the number of additional states due to non-Abelian QH exchange. Remarkably, the values in Table V differ from those[19] of Moore-Read state (e.g., $\eta-\eta_a=0$ for $K=2$, and $\eta-\eta_a=0,0,1,0,1,1,\ldots$ for $K=4$ and $\Lambda-L=0,1,2,\ldots$, respectively).

Let us summarize discussion of the $E=0$ ground state of $V=\delta_3(\mathcal{R})$ at $\nu=\frac{1}{5}$: (i) the fact that the largest angular momentum Λ predicted from CF transformation (1) agrees with the largest L of the $E=0$ states for nearly (see below) every combination of N and $2\ell\geq 5N-9$ supports this transformation and the QH picture of the $E=0$ subspace. (ii) The (unexplained) exception is the lack of $E=0$ states at $2\ell=5N-8$ in the CF picture corresponding to a single QH at $L=\frac{1}{2}N$. (iii) The fact that $\mathcal{D}>\mathcal{D}_a$ for $K\geq 2$ is a sign of non-Abelian statistics of the QHs. (iv) The dependences of $\eta-\eta_a$ on $\mathcal{R}^*$ for different numbers K are an important characteristic of the non-Abelian QHs; here, they allow for distinction from the statistics of Moore-Read QHs.

Let us return to the $\nu=\frac{1}{3}$ state of CFs. On a sphere, it is represented by a series of $L=0$ ground states at $2\ell=3N-7$ (distinct from $3N-3$ of the Laughlin state of individual fermions and from $3N-5$ of the hypothetical Laughlin-correlated states of $\mathcal{R}=1$ fermion pairs).[18]

Transformation (1) can be naturally extended to

$$2\ell^* = |2\ell - p(N-2)|, \tag{2}$$

with an arbitrary number p of flux quanta attached to each CF. In contrast to the original CF picture of Jain,[6] odd values of p must also be admitted due to pairing (path of a given particle can only encircle a whole other pair). Let us consider an arbitrary number $|n|$ of completely filled CF* shells, with the effective magnetic field pointing either in the same or in the opposite direction to the fictitious external field giving rise to the degeneracy of CF-LL_1. The latter case, corresponding to $2\ell<p(N-2)$, will be conveniently distinguished by a negative sign of n.

The filling of CF* shells yields a family of CF states at

$$2\ell = (p+n^{-1})N-(2p+n), \tag{3}$$

extrapolating to $\nu\equiv\lim(N/2\ell)=(p+n^{-1})^{-1}$ on a plane (some fractions ν result for two combinations of p and n). By construction, Eq. (3) includes the $(p,n)=(4,1)$ zero-energy state at $2\ell=5N-9$. Remarkably, the $\nu=\frac{1}{3}$ state at $2\ell=3N-7$ also

TABLE V. Difference between numbers η of zero-energy angular momentum multiplets L from Table IV and numbers η_a predicted for corresponding systems of K Abelian quasiholes.

K	2									3								4							
$\Lambda-L$	0	1	2	3	4	5	6	7	8	0	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7
$\eta-\eta_a$							1		1					1	1	2	2			1		2	2	5	5

emerges as $(4,-1)$, while $(1,1)$ reproduces another familiar $\nu=\frac{1}{2}$ series at $2\ell=2N-3$.

Different only by the sign of n (direction of effective magnetic field), the $\nu=\frac{1}{3}$ state $(4,-1)$ might have similar CF correlations to the $\nu=\frac{1}{5}$ state (4,1). This connection suggests non-Abelian QP excitations at $\nu_e=\frac{4}{11}$. Unfortunately, the numerical spectra of $V=\delta_3$ for $N\leq 12$ at $2\ell\approx 3N-7$ (not shown) are not conclusive: they contain low-energy QP bands predicted by transformation (2) but show no obvious sign of non-Abelian statistics.

To check which of the (p,n) states of Eq. (3) actually occurs for the interacting CFs, we computed ground-state energies of $N\leq 10$ fermions interacting through $V=\delta_3$ or through a more realistic CF pseudopotential of Lee *et al.*[17] as a function of 2ℓ. Results for $N=10$ are shown in Fig. 2. The largest excitation gaps Δ occur for $(p,n)=(4,-1)$ and (1,1), corresponding to the known[13] FQH states at $\nu_e=\frac{4}{11}$ and $\frac{3}{8}$. Sizable gap is also found for $(4,-2)$, suggesting a (so far undetected) FQH state at $\nu_e=\frac{9}{25}$. Other states, including the parent state (4,1), show only marginal incompressibility.

In conclusion, we postulate emergence of non-Abelian statistics in the lowest LL, in a family of second-generation liquids of correlated CFs. The argument involves: (i) identification of zero-energy state of a model CF-CF interaction; (ii) definition of flux attachment scheme converting this state into a filled shell; (iii) demonstration of non-Abelian statistics of its QHs; and (iv) construction of the hierarchy of IQLs, including known FQH states at $\nu_e=\frac{4}{11}$ and $\frac{3}{8}$ and a new state at $\nu_e=\frac{9}{25}$. These concepts were confirmed by exact diagonalization studies.

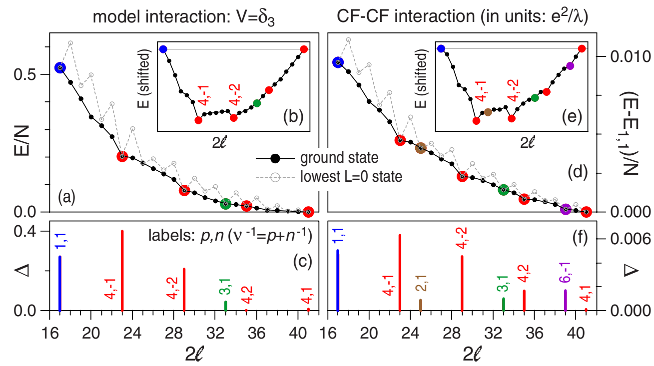

FIG. 2. (Color online) (a) Ground-state energy per particle E/N (also, lowest energy at $L=0$) of $N=10$ particles with model interaction $V=\delta_3$ on a sphere as a function of shell angular momentum ℓ. (b) Same as (a) but energy E shifted by const$\times\ell$ so that $E_{1,1}=E_{4,1}$ (to emphasize cusps). (c) Excitation gaps of $L=0$ ground states. (d)–(f) Same as (a)–(c) but for pseudopotential in CF-LL$_1$ taken from Ref. 17.

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