

JUSTIFICATION FOR THE COMPOSITE FERMION PICTURE

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The mean field (MF) composite Fermion (CF) picture successfully predicts the low-lying bands of states of fractional quantum Hall systems. This success cannot be attributed to the originally proposed cancellation between Coulomb and Chern–Simons interactions beyond the mean field and solely depends on the short range of the repulsive Coulomb pseudopotential in the lowest Landau level (LL). The class of pseudopotentials is defined for which the MFCF picture can be applied. The success or failure of the MFCF picture in various systems (electrons in the lowest and excited LL’s, Laughlin quasiparticles) is explained.

1. Introduction

The quantum Hall effect (QHE)^{1,2} is the quantization of Hall conductance of a two-dimensional electron gas (2DEG) in high magnetic fields that occurs at certain fillings of the macroscopically degenerate single-electron Landau levels (LL’s). The LL filling is defined by the filling factor ν equal to the number of electrons divided by the LL degeneracy, which is proportional to the magnetic field and the physical area occupied by the 2DEG. The series of values of ν at which the QHE is observed contains small integers and simple, almost exclusively odd-denominator fractions such as $\nu = \frac{1}{3}, \frac{2}{5}$, etc. The series is universal for all samples, which means that the occurrence of QHE at a particular value of ν depends on the quality of the sample, temperature, or other similar conditions, but not on the material parameters such as lattice composition. Moreover, the observed values of ν are “exact” in a sense that poor sample quality (e.g., lattice imperfections) can destroy QHE at some or all of the values of ν , but cannot shift these values.

The quantization of Hall conductance is always accompanied by a rapid drop of longitudinal conductance, and both effects signal the appearance of (incompressible) nondegenerate many-body ground states (GS’s) in the spectrum of the 2DEG, separated from the continuum of excited states by

a finite gap. At integer $\nu = 1, 2 \dots$ (IQHE), the origin of incompressibility is the single particle cyclotron gap between the LL's. On the other hand, at fractional $\nu = 1/3, 1/5, 2/5 \dots$ (FQHE) electrons partially fill a degenerate (lowest) LL and the formation of incompressible GS's is a complicated many-body phenomenon. The gap and resulting incompressibility are entirely due to electron-electron (Coulomb) interactions and reveal the unique properties of this interaction within the lowest LL.^{3,4}

In this note we shall concentrate on the latter effect. We will apply the pseudopotential formalism^{5,6} to the FQH systems, and show that the form of the pseudopotential $V(L')$ [pair energy vs. pair angular momentum] in the lowest LL rather than of the interaction potential $V(r)$, is responsible for incompressibility of the FQH states. The idea of fractional parentage⁷ will be used to characterize many-body states by the ability of electrons to avoid pair states with largest repulsion. The condition on the form of $V(L')$ necessary for the occurrence of FQH states will be given, which defines the short-range repulsive (SRR) pseudopotentials to which MFCF picture can be applied. As an example, we explain the success or failure of MFCF predictions for the electrons in the lowest and excited LL's and for Laughlin quasiparticles (QP's) in hierarchy picture of FQH states.^{8,9}

2. Theoretical concepts for FQHE

2.1. Single-electron states and Laughlin wavefunction

The Hamiltonian for an electron confined to the x - y plane in the presence of a perpendicular magnetic field B is

$$H_0 = \frac{1}{2\mu} \left(\vec{p} + \frac{e}{c} \vec{A} \right)^2. \quad (1)$$

Here μ is the effective mass, $\vec{p} = (p_x, p_y, 0)$ is the momentum operator and $\vec{A}(x, y)$ is the vector potential (whose curl gives B). For the "symmetric gauge," $\vec{A} = \frac{1}{2}B(-y, x, 0)$, the single particle eigenfunctions¹⁰ are of the form $\psi_{nm}(r, \theta) = e^{-im\theta} u_{nm}(r)$, and the eigenvalues are given by

$$E_{nm} = \frac{1}{2} \hbar \omega_c (2n + 1 + |m| - m). \quad (2)$$

In these equations, $n = 0, 1, 2, \dots$, and $m = 0, \pm 1, \pm 2, \dots$. The lowest energy states (lowest Landau level) have $n = 0$, $m = 0, 1, 2, \dots$, and energy $E_{0m} = \hbar \omega_c / 2$. It is convenient to introduce a complex coordinate $z = r e^{-i\theta} = x - iy$, and to write the lowest Landau level wavefunctions as $\psi_{0,-m} = N_m z^m \exp(-|z|^2/4)$, where N_m is a normalization constant, and m can take on any non-negative integral value. In this expression we

have used the magnetic length $\lambda = (\hbar c/eB)^{1/2}$ as the unit of length. The function $|\psi_{0,-m}|^2$ has its maximum at a radius r_m which is proportional to $m^{1/2}$. All single particle states from a given Landau level are degenerate, and separated in energy from neighboring levels by $\hbar\omega_c$.

If m is restricted to being less than some maximum value, N_L , chosen so that the system has a “finite radial range,” then the allowed m values are $0, 1, 2, \dots, N_L - 1$. The value of N_L is equal to the flux through the sample, $B \cdot A$ (where A is the area), divided by the quantum of flux $\phi_0 = hc/e$. The filling factor ν is defined as the ratio of the number of electrons, N , to N_L . An infinitesimal decrease in the area A when ν has an integral value requires promotion of an electron across the gap $\hbar\omega_c$ to the first unoccupied level, making the system incompressible.

In order to construct a many electron wavefunction corresponding to filling factor $\nu = 1$, the product function which places one electron in each of the N_L orbitals $\psi_{0,-m}$ ($m = 0, 1, \dots, N_L - 1$) must be antisymmetrized. This can be done with the aid of a Slater determinant

$$\Psi(z_1, z_2, \dots, z_N) \propto \begin{vmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_N \\ z_1^2 & z_2^2 & \dots & z_N^2 \\ \vdots & \vdots & & \vdots \\ z_1^{N_L-1} & z_2^{N_L-1} & \dots & z_N^{N_L-1} \end{vmatrix} e^{-\sum_k |z_k|^2/4}. \quad (3)$$

The determinant in Eq. (3) is the well-known Vandemonde determinant. It is not difficult to show that it is equal to $\prod_{i<j} (z_i - z_j)$. Of course, $N_L = N$ (since each of the N_L orbitals is occupied by one electron) and $\nu = 1$.

Laughlin noticed that if the factor $(z_i - z_j)$ of Vandemonde determinant was replaced by $(z_i - z_j)^m$, where m was an odd integer, the wavefunction

$$\Psi_m(z_1, z_2, \dots, z_N) \propto \prod_{i<j} (z_i - z_j)^m e^{-\sum_k |z_k|^2/4} \quad (4)$$

would be antisymmetric, keep the electrons further apart (and therefore reduce repulsion), and correspond to a filling factor $\nu = m^{-1}$. This results because the highest power of the orbital index entering Ψ_m is $N_L - 1 = m(N - 1)$ giving $\nu = N/N_L = m^{-1}$ in the limit of large systems. The additional factor $\prod_{i<j} (z_i - z_j)^{m-1}$ multiplying $\Psi_{m=1}(z_1, z_2, \dots, z_N)$ is the Jastrow factor which accounts for correlations.

2.2. Laughlin quasiparticles and Haldane hierarchy

The elementary charged excitations of the Laughlin $\nu = (2p + 1)^{-1}$ GS are Laughlin quasiparticles (QP's) corresponding to a vortex (for the quasihole,

QH) or anti-vortex (for the quasielectron, QE) at an arbitrary point z_0 , and described by the following wavefunctions,

$$\Psi_{\text{QH}} \propto (z - z_0)\Psi_m \quad \text{and} \quad \Psi_{\text{QE}} \propto \frac{\partial}{\partial(z - z_0)}\Psi_m. \quad (5)$$

Laughlin QP's carry fractional electric charge $\pm e/(2p + 1)$ and obey fractional statistics (although in some situations they can be conveniently treated as either fermions or bosons thanks to a statistics transformation valid in two dimensions). Being charged particles (of the finite size of the order of the magnetic length λ) moving in the magnetic field, Laughlin QP form (quasi)LL's similar to those of electrons, except for the m -times lower degeneracy due to their reduced charge. They are also naturally expected to interact with one another via normal, charge-charge Coulomb forces.

As an extension of Laughlin's idea, Haldane,⁸ and others^{9,11} proposed that in analogy to electrons, Laughlin QP's must form Laughlin-like incompressible states of their own. According to this idea, each Laughlin state of electrons would stand atop entire family of so-called "daughter" Laughlin state of its QE's and QH's. And on the following level, each daughter state would have its own family of daughter states, and so on. This construction results in entire family of incompressible states that corresponds to many more fractional filling factors ν at which incompressibility and in result also the FQHE are expected in the underlying 2DEG. For example, the $\nu = \frac{2}{5}$ state can be interpreted as $\nu = \frac{1}{3}$ QE daughter state of the parent $\nu = \frac{1}{3}$ electron state. In fact, all odd-denominator fractions $\nu = p/q$ can be generated in this way, which brings us to the major problem of the (original) concept of hierarchy. On one hand, the hierarchy predicts too many fractions (only a finite number of fractions are observed experimentally, and it seems evident that FQHE will not occur at most of the other predicted fractions regardless of the experimental conditions^{2,12,13}). On the other hand, the hierarchy model gives no apparent connection between the stability of a given state and its position in the hierarchy (explanation of some of the easily experimentally observed FQH states requires introducing many generations of QP's). As we shall explain later, these problems of the hierarchy model resulted from an erroneous assumption that, being charged particles, Laughlin QP's will form Laughlin states at each Laughlin filling factor, $\nu = (2p + 1)^{-1}$. With the knowledge of the form of QP-QP interactions, one can eliminate "false" daughter states from the hierarchy and reach an agreement with the experimental observation. This makes the Laughlin-Haldane theory the only microscopic theory of the FQHE.

2.3. Jain composite Fermion model

Independently of the Laughlin–Haldane model, from the similar energy spectra of the FQH and IQH systems one can expect that some kind of effective, charged particle-like excitations may form in the interacting 2DEG. These excitations would be the relevant charge carriers near $\nu = \frac{1}{3}$ and they would fill exactly their quasi-LL's at precisely this value, giving rise to incompressibility. This idea leads to the composite Fermion (CF) picture.^{14,15}

In the mean field (MF) CF picture, in a 2DEG of density n at a strong magnetic field B , each electron is assumed to bind an even number $2p$ of magnetic flux quanta $\phi_0 = hc/e$ (in form of an infinitely thin flux tube) forming a CF. Because of the Pauli exclusion principle, the magnetic field confined into a flux tube within one CF has no effect on the motion of other CF's, and the average effective magnetic field B^* seen by CF's is reduced, $B^* = B - 2p\phi_0 n$. Because $B^*\nu^* = B\nu = n\phi_0$, the relation between the electron and CF filling factors is

$$(\nu^*)^{-1} = \nu^{-1} - 2p. \quad (6)$$

Since the low band of energy levels of the original (interacting) 2DEG has similar structure to that of the noninteracting CF's in a uniform effective field B^* , it was proposed¹⁴ that the Coulomb charge-charge and Chern–Simons (CS) charge-flux interactions beyond the MF largely cancel one another, and the original strongly interacting system of electrons is converted into one of weakly interacting CF's. Consequently, the FQHE of electrons was interpreted as the IQHE of CF's.

Although the MFCF picture correctly predicts the structure of low-energy spectra of FQH systems, the energy scale it uses (the CF cyclotron energy $\hbar\omega_c^*$) is totally irrelevant. Moreover, since the characteristic energies of CS ($\hbar\omega_c^* \propto B$) and Coulomb ($e^2/\lambda \propto \sqrt{B}$, where λ is the magnetic length) interactions between fluctuations beyond MF scale differently with the magnetic field, the reason for its success cannot be found in originally suggested cancellation between those interactions. Since the MFCF picture is commonly used to interpret various numerical and experimental results, it is important to understand why and under what conditions it is correct.

3. Numerical exact diagonalization studies

Because of the LL degeneracy, the electron-electron interaction in the FQH states cannot be treated perturbatively, and the exact (numerical) diagonalization techniques have been commonly used in their study. In order to model an infinite 2DEG by a finite (small) system that can be handled

numerically, it is very convenient to confine N electrons to a surface of a (Haldane) sphere of radius R , with the normal magnetic field B produced by a magnetic monopole of integer strength $2S$ (total flux of $4\pi BR^2 = 2S\phi_0$) in the center.⁸ The obvious advantages of such geometry is the absence of an edge and preserving full 2D symmetry of a 2DEG (good quantum numbers are the total angular momentum L and its projection M). The numerical experiments in this geometry have shown that even relatively small systems that can be solved exactly on a computer behave in many ways like an infinite 2DEG, and a number of parameters of a 2DEG (e.g. excitation energies) can be obtained from such small scale calculations.

The single particle states on a Haldane sphere (monopole harmonics) are labeled by angular momentum l and its projection m .¹⁶ The energies, $\varepsilon_l = \hbar\omega_c[l(l+1) - S^2]/2S$, fall into degenerate shells and the n th shell ($n = l - |S| = 0, 1, \dots$) corresponds to the n th LL. For the FQH states at filling factor $\nu < 1$, only the lowest, spin polarized LL need be considered.

The object of numerical studies is to diagonalize the electron-electron interaction Hamiltonian H in the space of degenerate antisymmetric N electron states of a given (lowest) LL. Although matrix H is easily block diagonalized into blocks with specified M , the exact diagonalization becomes difficult (matrix dimension over 10^6) for $N > 10$ and $2S > 27$ ($\nu = 1/3$).⁶ Typical results for ten electrons at filling factors near $\nu = 1/3$ are presented in Fig. 1. Energy E , plotted as a function of L in the magnetic units, includes shift $-(Ne)^2/2R$ due to charge compensating background. There is always one or more L multiplets (marked with open circles) forming a low-energy band separated from the continuum by a gap. If the lowest band consists of a single $L = 0$ GS (Fig. 1d), it is expected to be incompressible in the thermodynamic limit (for $N \rightarrow \infty$ at the same ν) and an infinite 2DEG at this filling factor is expected to exhibit the FQHE.

The MFCF interpretation of the spectra in Fig. 1 is the following. The effective magnetic monopole strength seen by CF's is^{14,6}

$$2S^* = 2S - 2p(N - 1), \quad (7)$$

and the angular momenta of lowest CF shells (CF LL's) are $l_n^* = |S^*| + n$.¹⁷ At $2S = 27$, $l_0^* = 9/2$ and ten CF's fill completely the lowest CF shell ($L = 0$ and $\nu^* = 1$). The excitations of the $\nu^* = 1$ CF GS involve an excitation of at least one CF to a higher CF LL, and thus (if the CF-CF interaction is weak on the scale of $\hbar\omega_c^*$) the $\nu^* = 1$ GS is incompressible and so is Laughlin³ $\nu = 1/3$ GS of underlying electrons. The lowest lying excited states contain a pair of QP's: a quasihole (QH) with $l_{\text{QH}} = l_0^* = 9/2$ in the lowest CF LL and a quasielectron (QE) with $l_{\text{QE}} = l_1^* = 11/2$ in the first

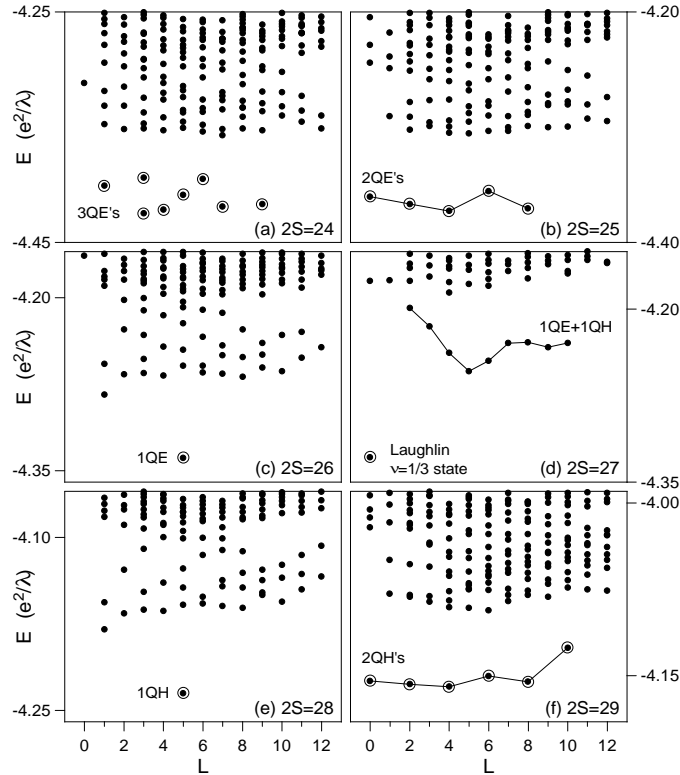


Figure 1. Energy spectra of ten electrons in the lowest LL at the monopole strength $2S$ between 24 and 29. Open circles mark lowest energy bands with fewest CF QP's.

excited one. The allowed angular momenta of such pair are $L = 1, 2, \dots, 10$. The $L = 1$ state usually has high energy and the states with $L \geq 2$ form a well defined band with a magnetoroton minimum at a finite value of L . The lowest CF states at $2S = 26$ and 28 contain a single QE and a single QH, respectively (in the $\nu^* = 1$ CF state, i.e. the $\nu = 1/3$ electron state), both with $l_{QP} = 5$, and the excited states will contain additional QE-QH pairs. At $2S = 25$ and 29 the lowest bands correspond to a pair of QP's, and the values of energy within those bands define the QP-QP interaction pseudopotential V_{QP} . At $2S = 25$ there are two QE's each with $l_{QE} = 9/2$ and the allowed angular momenta (of two identical Fermions) are $L = 0, 2, 4, 6, 8$, while at $2S = 29$ there are two QH's each with $l_{QH} = 11/2$ and $L = 0, 2, 4, 6, 8, 10$. Finally, at $2S = 24$, the lowest band contains three QE's each with $l_{QE} = 4$ and $L = 1, 3^2, 4, 5, 6, 7, 9$.

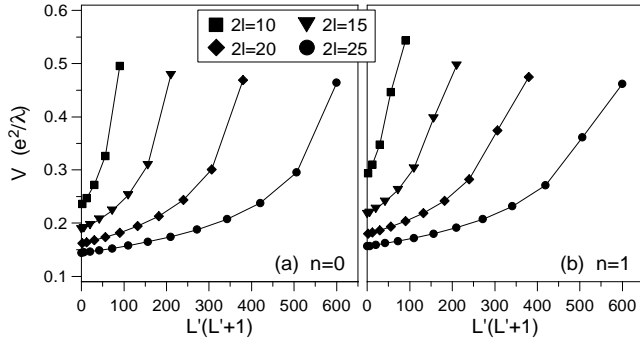


Figure 2. Pseudopotentials V of the Coulomb interaction in the lowest (a), and first excited LL (b) as a function of squared pair angular momentum $L'(L' + 1)$. Different symbols mark data for different $S = l + n$.

4. Pseudopotential and fractional grandparentage

The two body interaction Hamiltonian H can be expressed as

$$\hat{H} = \sum_{i < j} \sum_{L'} V(L') \hat{\mathbf{P}}_{ij}(L'), \quad (8)$$

where $V(L')$ is the interaction pseudopotential⁵ and $\hat{\mathbf{P}}_{ij}(L')$ projects onto the subspace with angular momentum of pair ij equal to L' . For electrons confined to a LL, L' measures the average squared distance d^2 ,⁶

$$\frac{\hat{d}^2}{R^2} = 2 + \frac{S^2}{l(l+1)} \left(2 - \frac{\hat{L}^2}{l(l+1)} \right), \quad (9)$$

and larger L' corresponds to smaller separation. Due to the confinement of electrons to one (lowest) LL, interaction potential $V(r)$ enters Hamiltonian H only through a small number of pseudopotential parameters $V(2l - \mathbf{R})$, where \mathbf{R} , relative pair angular momentum, is an odd integer.

In Fig. 2 we compare Coulomb pseudopotentials $V(L')$ calculated for a pair of electrons on the Haldane sphere each with $l = 5, 15/2, 10,$ and $25/2$, in the lowest and first excited LL. For the reason that will become clear later, $V(L')$ is plotted as a function of $L'(L' + 1)$. All pseudopotentials in Fig. 2 increase with increasing L' . If $V(L')$ increased very quickly with increasing L' (we define ideal short-range repulsion, SRR, as: $dV_{\text{SR}}/dL' \gg 0$ and $d^2V_{\text{SR}}/dL'^2 \gg 0$), the low-lying many-body states would be the ones maximally avoiding pair states with largest L' .^{5,6} At filling factor $\nu = 1/m$ (m is odd) the many-body Hilbert space contains exactly one multiplet in which all pairs completely avoid states with $L' > 2l - m$. This multiplet is the $L = 0$ incompressible Laughlin state³ and it is an exact GS of V_{SRR} .

The ability of electrons in a given many-body state to avoid strongly repulsive pair states can be conveniently described using the idea of fractional parentage.^{6,7} An antisymmetric state $|l^N, L\alpha\rangle$ of N electrons each with angular momentum l that are combined to give total angular momentum L can be written as

$$|l^N, L\alpha\rangle = \sum_{L'} \sum_{L''\alpha''} G_{L\alpha}^{L''\alpha''}(L') |l^2, L'; l^{N-2}, L''\alpha''; L\rangle. \quad (10)$$

Here, $|l^2, L'; l^{N-2}, L''\alpha''; L\rangle$ denote product states in which $l_1 = l_2 = l$ are added to obtain L' , $l_3 = l_4 = \dots = l_N = l$ are added to obtain L'' (different L'' multiplets are distinguished by a label α''), and finally L' is added to L'' to obtain L . The state $|l^N, L\alpha\rangle$ is totally antisymmetric, and states $|l^2, L'; l^{N-2}, L''\alpha''; L\rangle$ are antisymmetric under interchange of particles 1 and 2, and under interchange of any pair of particles 3, 4, \dots N . The factor $G_{L\alpha}^{L''\alpha''}(L')$ is called the coefficient of fractional grandparentage (CFGP). The two-body interaction matrix element is expressed as

$$\langle l^N, L\alpha | V | l^N, L\beta \rangle = \frac{N(N-1)}{2} \sum_{L'; L''\alpha''} G_{L\alpha}^{L''\alpha''}(L') G_{L\beta}^{L''\alpha''}(L') V(L'), \quad (11)$$

and expectation value of energy is

$$E_\alpha(L) = \frac{N(N-1)}{2} \sum_{L'} \mathbf{G}_{L\alpha}(L') V(L'), \quad (12)$$

where the coefficient

$$\mathbf{G}_{L\alpha}(L') = \sum_{L''\alpha''} \left| G_{L\alpha}^{L''\alpha''}(L') \right|^2 \quad (13)$$

gives the probability that pair ij is in the state with L' .

5. Energy spectra of short-range repulsive pseudopotentials

The very good description of actual GS's of a 2DEG at fillings $\nu = 1/m$ by the Laughlin wavefunction (overlaps typically larger than 0.99) and the success of the MFCF picture at $\nu < 1$ both rely on the fact that pseudopotential of Coulomb repulsion in the lowest LL falls into the same class of SRR pseudopotentials as V_{SRR} . Due to a huge difference between all parameters $V_{\text{SRR}}(L')$, the corresponding many-body Hamiltonian has the following hidden symmetry: the Hilbert space \mathbf{H} contains eigensubspaces \mathbf{H}_p of states with $\mathbf{G}(L') = 0$ for $L' > 2(l-p)$, i.e. with $L' < 2(l-p)$. Hence, \mathbf{H} splits into subspaces $\tilde{\mathbf{H}}_p = \mathbf{H}_p \setminus \mathbf{H}_{p+1}$, containing states that do

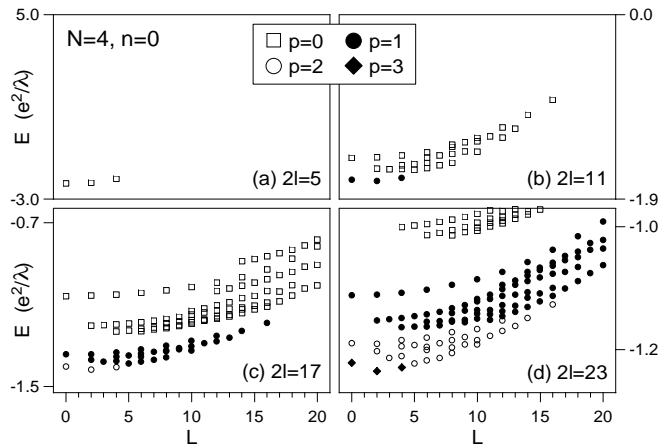


Figure 3. Energy spectra of four electrons in the lowest LL each with angular momentum $l = 5/2$ (a), $11/2$ (b), $17/2$ (c), and $23/2$ (d). Different subspaces \mathbf{H}_p are marked with squares ($p = 0$), full ($p = 1$) and open circles ($p = 2$), and diamonds ($p = 3$).

not have grandparentage from $L' > 2(l - p)$, but have some grandparentage from $L' = 2(l - p) - 1$,

$$\mathbf{H} = \tilde{\mathbf{H}}_0 \oplus \tilde{\mathbf{H}}_1 \oplus \tilde{\mathbf{H}}_2 \oplus \dots \quad (14)$$

The subspace $\tilde{\mathbf{H}}_p$ is not empty (some states with $L' < 2(l - p)$ can be constructed) at filling factors $\nu \leq (2p + 1)^{-1}$. Since the energy of states from each subspace $\tilde{\mathbf{H}}_p$ is measured on a different scale of $V(2(l - p) - 1)$, the energy spectrum splits into bands corresponding to those subspaces. The energy gap between the p th and $(p + 1)$ st bands is of the order of $V(2(l - p) - 1) - V(2(l - p - 1) - 1)$, and hence the largest gap is that between the 0th band and the 1st band, the next largest is that between the 1st band and 2nd band, etc.

Fig. 3 demonstrates on the example of four electrons to what extent this hidden symmetry holds for the Coulomb pseudopotential in the lowest LL. The subspaces \mathbf{H}_p are identified by calculating CFGP's of all states. They are not exact eigenspaces of the Coulomb interaction, but the mixing between different \mathbf{H}_p is weak and the coefficients $\mathbf{G}(L')$ for $L' > 2(l - p)$ are indeed much smaller in states marked with a given p than in all other states. For example, for $2l = 11$, $\mathbf{G}(10) < 0.003$ for states marked with full circles, and $\mathbf{G}(10) > 0.1$ for all other states (squares).

Note that the set of angular momentum multiplets which form subspace $\tilde{\mathbf{H}}_p$ of N electrons each with angular momentum l is always the same as the set of multiplets in subspace $\tilde{\mathbf{H}}_{p+1}$ of N electrons each with angular

momentum $l + (N - 1)$. When l is increased by $N - 1$, an additional band appears at high energy, but the structure of the low-energy part of the spectrum is completely unchanged. For example, all three allowed multiplets for $l = 5/2$ ($L = 0, 2$, and 4) form the lowest energy band for $l = 11/2, 17/2$, and $23/2$, where they span the $\tilde{\mathbf{H}}_1, \tilde{\mathbf{H}}_2$ and $\tilde{\mathbf{H}}_3$ subspace, respectively. Similarly, the first excited band for $l = 11/2$ is repeated for $l = 17/2$ and $23/2$, where it corresponds to $\tilde{\mathbf{H}}_1$ and $\tilde{\mathbf{H}}_2$ subspace.

Let us stress that the fact that identical sets of multiplets occur in subspace $\tilde{\mathbf{H}}_p$ for a given l and in subspace $\tilde{\mathbf{H}}_{q+1}$ for l replaced by $l + (N - 1)$, does not depend on the form of interaction, and follows solely from the rules of addition of angular momenta of identical Fermions. However, if the interaction pseudopotential has SRR, then: (i) $\tilde{\mathbf{H}}_p$ are interaction eigensubspaces; (ii) energy bands corresponding to $\tilde{\mathbf{H}}_p$ with higher p lie below those of lower p ; (iii) spacing between neighboring bands is governed by a difference between appropriate pseudopotential coefficients; and (iv) wavefunctions and structure of energy levels within each band are insensitive to the details of interaction. Replacing V_{SRR} by a pseudopotential that increases more slowly with increasing L' leads to: (v) coupling between subspaces $\tilde{\mathbf{H}}_p$; (vi) mixing, overlap, or even order reversal of bands; (vii) deviation of wavefunctions and the structure of energy levels within bands from those of the hard core repulsion (and thus their dependence on details of the interaction pseudopotential). The numerical calculations for the Coulomb pseudopotential in the lowest LL show (to a large extent) all SRR properties (i)–(iv), and virtually no effects (v)–(vii), characteristic of ‘non-SRR’ pseudopotentials.

The reoccurrence of L multiplets forming the low-energy band when l is replaced by $l \pm (N - 1)$ has the following crucial implication. In the lowest LL, the lowest energy (p th) band of the N electron spectrum at the monopole strength $2S$ contains L multiplets which are all the allowed N electron multiplets at $2S - 2p(N - 1)$. But $2S - 2p(N - 1)$ is just $2S^*$, the effective monopole strength of CF’s! The MFCs transformation which binds $2p$ fluxes (vortices) to each electron selects the same L multiplets from the entire spectrum as does the introduction of a hard core, which forbids a pair of electrons to be in a state with $L' > 2(l - p)$.

6. Definition of short-range repulsive pseudopotential

A useful operator identity relates total (L) and pair (\hat{L}_{ij}) angular momenta⁶

$$\sum_{i < j} \hat{L}_{ij}^2 = \hat{L}^2 + N(N - 2) \hat{l}^2. \quad (15)$$

It implies that interaction given by a pseudopotential $V(L')$ that is linear in \hat{L}'^2 (e.g. the harmonic repulsion within each LL) is degenerate in each L subspace and its energy is a linear function of $L(L+1)$. The many-body GS has the lowest available L while the maximum L corresponds to the largest energy. Note that this result is opposite to the Hund rule valid for spherical harmonics, due to the opposite behavior of $V(L')$ for the FQH ($n=0$ and $l=S$) and atomic ($S=0$ and $l=n$) systems.

Deviations of $V(L')$ from a linear function of $L'(L'+1)$ lead to the level repulsion within each L subspace, and the GS is no longer necessarily the state with minimum L . Rather, it is the state at a low L whose multiplicity N_L (number of different L multiplets) is large. It is interesting to observe that the L subspaces with relatively high N_L coincide with the MFCCF prediction. In particular, for a given N , they reoccur at the same L 's when l is replaced by $l \pm (N-1)$, and the set of allowed L 's at a given l is always a subset of the set at $l + (N-1)$.

As we said earlier, if $V(L')$ has short range, the lowest energy states within each L subspace are those maximally avoiding large L' , and the lowest band (separated from higher states by a gap) contains states in which a number of largest values of L' is avoided altogether. This property is valid for all pseudopotentials which increase more quickly than linearly as a function of $L'(L'+1)$. For $V_\beta(L') = [L'(L'+1)]^\beta$, exponent $\beta > 1$ defines the class of SRR pseudopotentials, to which the MFCCF picture can be applied. Within this class, the structure of low-lying energy spectrum and the corresponding wavefunctions very weakly depend on β and converge to those of V_{SRR} for $\beta \rightarrow \infty$.

The extension of the SRR definition to $V(L')$ that are not strictly in the form of $V_\beta(L')$ is straightforward. If $V(L') > V(2l-m)$ for $L' > 2l-m$ and $V(L') < V(2l-m)$ for $L' < 2l-m$ and $V(L')$ increases more quickly than linearly as a function of $L'(L'+1)$ in the vicinity of $L' = 2l-m$, then pseudopotential $V(L')$ behaves like SRR at filling factors near $\nu = 1/m$.

7. Application to various interacting systems

It follows from Fig. 2a that the Coulomb pseudopotential in the lowest LL satisfies the SRR condition in the entire range of L' ; this is what validates the MFCCF picture for filling factors $\nu \leq 1$. However, in a higher, n th LL this is only true for $L' < 2(l-n) - 1$ (see Fig. 2b for $n=1$) and the MFCCF picture is valid only for ν_n (filling factor in the n th LL) around and below $(2n+3)^{-1}$. Indeed, the MFCCF features in the ten electron energy spectra around $\nu = 1/3$ (in Fig. 1) are absent for the same fillings of the $n=1$ LL.⁶

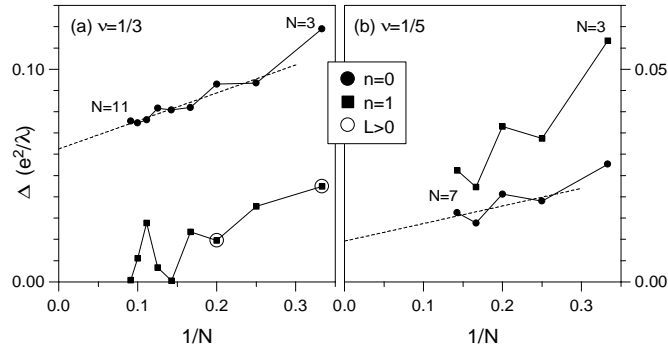


Figure 4. Excitation gap Δ as a function of inverse electron number $1/N$ for filling factors $\nu = 1/3$ (a) and $1/5$ (b) in the $n = 0$ (dots) and $n = 1$ (squares) LL's. Open circles mark degenerate ground states ($L > 0$).

One consequence of this is that the MFCF picture or Laughlin-like wavefunction cannot be used to describe the reported¹² incompressible state at $\nu = 2 + 1/3 = 7/3$ ($\nu_1 = 1/3$). The correlations in the $\nu = 7/3$ GS are different than at $\nu = 1/3$; the origin of (apparent) incompressibility cannot be attributed to the formation of a Laughlin-like $\nu_1 = 1/3$ state on top of the $\nu = 2$ state and connection between the excitation gap and the pseudopotential parameters is different. This is clearly visible in the dependence of the excitation gap Δ on the electron number N , plotted in Fig. 4 for $\nu = 1/3$ and $1/5$ fillings of the lowest and first excited LL. The gaps for $\nu = 1/5$ behave similarly as a function of N in both LL's, while it is not even possible to make a conclusive statement about degeneracy or incompressibility of the $\nu = 7/3$ state based on data for up to 11 electrons.

The SRR criterion can be applied to the QP pseudopotentials to understand why QP's do not form incompressible states at all Laughlin filling factors $\nu_{QP} = 1/m$ in the hierarchy picture^{8,9} of FQH states. Lines in Fig. 1b and 1f mark V_{QE} and V_{QH} for the Laughlin $\nu = 1/3$ state of ten electrons. From similar calculations for different N one can deduce the behavior of the QP pseudopotentials in the $N \rightarrow \infty$ limit. Such analysis leads to a surprising conclusion that the SRR character characteristic of the electron-electron pseudopotential in the lowest LL does not generally hold for the QP pseudopotentials.^{18,19} In Fig. 5 we show the data for QE's and QH's in Laughlin $\nu = 1/3$ (data for $N \leq 8$ was published before¹⁹) and $\nu = 1/5$ states. The plotted energy is given in units of e^2/λ where λ is the magnetic length in the parent state. Different symbols mark pseudopotentials obtained in diagonalization of N electron systems with different N

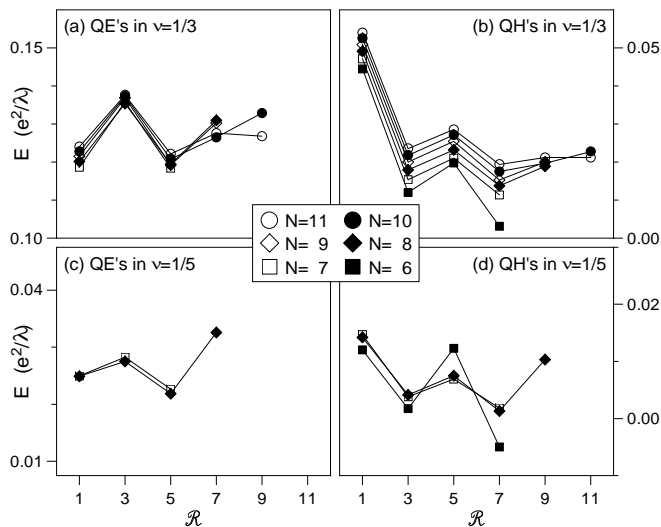


Figure 5. Energies of a pair of quasielectrons (left) and quasiholes (right) in Laughlin $\nu = 1/3$ (top) and $\nu = 1/5$ (bottom) states, as a function of relative pair angular momentum \mathbf{R} , obtained in diagonalization of N electrons.

and thus with different l_{QP}). Clearly, the QE and QH pseudopotentials are quite different and neither one decreases monotonically with increasing \mathbf{R} . On the other hand, the corresponding pseudopotentials in $\nu = 1/3$ and $1/5$ states look similar, only the energy scale is different. The convergence of energies at small \mathbf{R} obtained for larger N suggests that the maxima at $\mathbf{R} = 3$ for QE's and at $\mathbf{R} = 1$ and 5 for QH's, as well as the minima at $\mathbf{R} = 1$ and 5 for QE's and at $\mathbf{R} = 3$ and 7 for QH's, persist in the limit of large N (i.e. for an infinite system on a plane). Consequently, the only incompressible daughter states of Laughlin $\nu = 1/3$ and $1/5$ states are those with $\nu_{QE} = 1$ or $\nu_{QH} = 1/3$ (asterisks in Fig. 1) and (maybe) $\nu_{QE} = 1/5$ and $\nu_{QH} = 1/7$ (question marks in Fig. 1). It is also clear that no incompressible daughter states will form at e.g. $\nu = 4/11$ or $4/13$. Taking into account the behavior of involved QP pseudopotentials on all levels of hierarchy explains all observed odd-denominator FQH states and allows prediction of their relative stability (without using trial wavefunctions involving multiple LL's and projections onto the lowest LL needed in Jain's CF picture).

8. Conclusion

Using the pseudopotential formalism, we have described the FQH states in terms of the ability of electrons to avoid strongly repulsive pair states. We

have defined the class of SRR pseudopotentials leading to the formation of incompressible FQH states. We argue that the MFCF picture is justified for the SRR interactions and fails for others. The pseudopotentials of the Coulomb interaction in excited LL's and of Laughlin QP's in the $\nu = 1/3$ state are shown to belong to the SRR class only at certain filling factors.

Acknowledgment

AW and JJQ acknowledge partial support by the Materials Research Program of Basic Energy Sciences, US Department of Energy.

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