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Spin Orbitronics and Topological Properties of Nanostructures

Symmetry and Structural Properties of Condensed Matter

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Preface

This book includes some of the lectures presented at the 12th International School on Theoretical Physics "Symmetry and Structural Properties of Condensed Matter SSPCM 2016", which took place in Rzeszów (Poland) in September 2016. The SSPCM school is a traditional summer school, which is mostly focused on new ideas and modern methods in theoretical condensed matter physics. This is also a school for young scientists starting their carrier in physics as well as for graduate students. At the SSPCM schools they can meet well-known physicists from the United States, Japan, Russia, and European countries. The young researches can also present and discuss their own results during the School. Although the School is devoted mainly to theoretical physics, we traditionally invite lecturers working in computer physics and experiment as well. This demonstrates an important connection between the theory and experiment, and also gives young scientists a good chance to choose their own place in the modern physics.

Obviously, the main topics of the presented lecture notes reflect the recent interest of physicists. This includes the so-called spin-related phenomena in low-dimensional systems and nanostructures, spin dynamics, new materials and new states of matter, as well as some new methods based on topology and/or symmetry. All these topics appeared in the condensed matter physics in recent several years, and also paved the way to very interesting practical applications. The importance of spin-orbit interaction, which was the main element in many lectures, justifies the title of this book "Spin orbitronics", which became recently very popular in the condensed matter community, similarly like spintronics or spin caloritronics. It should be also emphasized that there are some expectations that these areas of activity will lead to a substantial advancement of modern nanotechnology as well as to the development of new quantum devices.

It is worth mentioning that the idea of the school, which is not focused on a single topic, makes the school in Rzeszów different from the other conferences and workshops. The main reason to organize such meetings is that good mathematical methods and novel physical ideas can be applied in dif-

Introduction to the Topic of Jack Polynomials in the Context of Fractional Quantum Hall Effect

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The Jack polynomials J^{α}_{λ} are a remarkable family of symmetric polynomials. They generalize many families of symmetric polynomials including: Schur polynomials, monomial symmetric polynomials, and elementary symmetric polynomials. The original development of Jack polynomials focused on the case of the real parameter $\alpha > 0$. This assures that the corresponding J^{α}_{λ} has no pole and is well defied. However for a specific values of $\alpha < 0$, Jack polynomials have no pole as well and reveal interesting properties, useful in physics of fractional quantum Hall effect. This solid state phenomenon involves essentially two-dimensional electrons in extremely low temperatures and strong magnetic field. Aim of this paper is to introduce the Jack polynomials in the context of fractional quantum Hall effect. We give a brief description of classical version of Hall effect and discuss both the integer and fractional quantum Hall effect. Then we introduce the basics of the theory of symmetric functions; in particular we define, describe, and construct the Jack polynomials. This knowledge is later applied and illustrated with numerical generation of the coefficients of "Jack states" (quantum Hall states related to the Jack polynomials). As an illustration we examine overlaps of the Jack state wave functions and the Coulomb ground states for different system sizes.

Keywords: Quantum Hall effect, Jack polynomials, Jack states, partitions.

1. Physics of quantum Hall effect

We start these notes with a brief introduction to the topic of quantum Hall effect. Firstly we discuss a classical version of the phenomenon, then follow with the integer quantum Hall effect. This includes analysis of one particle model and solutions to the stationary Schrödinger equation. Then, we focus on the fractional quantum Hall effect and introduce the standard tools used to study it, like: Haldane sphere, composite fermions, and trial wave function approach.

1.1. *Classical Hall effect*

The Hall effect was discovered by Edwin Herbert Hall in 1879. The phenomenon refers to an electric current passing through a cuboid conductor in a uniform magnetic field. For such systems the electric charge accumulates on the face of cuboid transverse to the applied magnetic field and the electric current.[1–3]

The Hall effect can be explained within frame of classical physics. Consider a flat cuboid conductor with sides a, b, c placed in a uniform magnetic field $\mathbf{B}$ parallel to the side c. Denote the magnitude of magnetic field by B. Assume there is a current parallel to a, and an average velocity of carriers equals $\mathbf{v_c}$. Then, on average, charge carrier of charge q experiences the magnetic force $\mathbf{F_m}$

$$\mathbf{F_m} = q\mathbf{v_c} \times \mathbf{B}. \tag{1}$$

This force pushes the charge carrier to one face of the cuboid. Hence one can observe charge density difference, and the y-component (parallel to the side b) of the electric field E_y builds up. Then, the electric carrier is driven by the electric force $\mathbf{F_e}$. Its y-component can be derived from a formula $(\mathbf{F_e})_y = qE_y$. For a density of carriers denoted ρ, the current $\mathbf{I}$ equals

$$\mathbf{I} = \mathbf{v_c}\rho(b \cdot c)q. \tag{2}$$

Since only the x-component of $\mathbf{I}$ is nonzero, one writes $I_x = v_c n(b \cdot c)q$, where v_c is a magnitude of $\mathbf{v_c}$. Then y-component of $\mathbf{F_m}$ equals

$$(\mathbf{F_m})_y = B\frac{I_x}{\rho(b \cdot c)}. \tag{3}$$

In an equilibrium net force equals zero therefore

$$0 = \mathbf{F_m} - \mathbf{F_e}. \tag{4}$$

What follows

$$B = \frac{I_x}{\rho(b \cdot c)} = qE_y. \tag{5}$$

For the Hall voltage defined as

$$V_H = E_y b, \tag{6}$$

one obtains

$$V_H = B\frac{I_x}{\rho q}. \tag{7}$$

The formula 7 shows that Hall voltage is proportional to the magnitude of the magnetic field. The Hall resistance is defined as $R_H = V_H/I_x$. Then

$$R_H = \frac{B}{\rho e c}. \tag{8}$$

1.2. *Integer quantum Hall effect*

The formula 7 suggests that Hall resistance is a linear function of magnetic field. However this relation is not fulfilled in two-dimensional, cold electron systems in high magnetic fields. In such systems, R_H is restricted to the specific values $R_H = \nu^{-1}\frac{h}{e^2}$, where h is Planck constant and e is electric charge.

In the case of integer quantum Hall effect (IQHE), $\nu \in \mathbb{N}_+$. Quantity ν is also known as a filling factor and is defined as

$$\nu = \frac{\rho}{\frac{B}{\phi_0}}. \tag{9}$$

where $\phi_0 = \frac{hc}{e}$ is a flux quantum. IQHE was first observed in 1980 at the High Magnetic Field Laboratory in Grenoble. In 1985 Klaus von Klitzing was awarded Nobel prize for its discovery. Due to the discrete values of the conductance, quantum Hall effect provides an extremely precise determination of the fine structure constant $\alpha = \frac{e^2}{4\pi\varepsilon_0 hc}$. IQHE is closely related to Landau quantization of cyclotron orbits of charged particles in magnetic field. Discrete energy levels of electrons are called the Landau levels (LL) and are highly degenerate.

1.2.1. *One particle model: Landau gauge*

In order to explain IQHE one can consider a model of one, non interacting particle of mass m and charge q, confined to an area $[0, L_x] \times [0, L_y]$ in an XY-plane.[2,3] Let the particle be exposed to a uniform magnetic field $\mathbf{B}$. Then the magnetic field enters the Hamiltonian H of such system through a vector potential $\mathbf{A}$ ($\mathbf{B} = \nabla \times \mathbf{A}$).

$$H = \frac{1}{2m}\left(\mathbf{p} - q\frac{\mathbf{A}}{c}\right)^2. \tag{10}$$

where $\mathbf{p} = (\mathbf{p_x}, \mathbf{p_y}, \mathbf{p_z})$ is a vector of the momentum operators and c is a speed of light. The square of vector is understand as a dot product of this vector with itself. In the Landau gauge

$$\mathbf{A} = (0, B\mathbf{x}, 0) \tag{11}$$

where $\mathbf{x}$ is an operator of x-component of a position. Then

$$H = \frac{1}{2m}\left(\mathbf{p_x}^2 + \left(\mathbf{p_y} - \frac{qB\mathbf{x}}{c}\right)^2\right). \tag{12}$$

One looks for solutions of an eigenvector problem ($Hv = e \cdot v$, $e \in \mathbb{R}$) in functions of a form:

$$v(x, y) = \exp(iky)\varphi(x). \tag{13}$$

When substituted to the equation 12, the problem reduces to one dimensional equation

$$e \cdot \varphi(x) = \left(-\frac{\hbar^2}{2m}\partial_x^2 + \frac{m\omega_c^2}{2}\left(x + k\ell_B^2\right)^2\right)\varphi(x), \tag{14}$$

for a cyclotron frequency $\omega_c^2 = \frac{qB}{mc}$ and a magnetic length $\ell_B = \sqrt{\frac{\hbar c}{qB}}$. Notice an equation 14 is equivalent to a shifted quantum harmonic oscillator.

An eigenvalue and an eigenvector are labelled by two numbers: k and n

$$H\psi_{k,n} = e_{k,n}\psi_{k,n}. \tag{15}$$

Then the eigenvalues are given by[4,5]

$$e_{k,n} = \left(n + \frac{1}{2}\right)\hbar\omega_c. \tag{16}$$

Notice there is no dependence in k in the equation 16. The eigenvectors $v_{k,n}$ equal

$$v_{k,n} = \exp(iky)H_n\left(\frac{x + k\ell_B^2}{l_B}\right)\exp\left(-\frac{1}{2l_B^2}(x + k\ell_B^2)^2\right), \tag{17}$$

where H_n are the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}. \tag{18}$$

The Hermite functions H_n, solutions to the quantum harmonic oscillator in natural units, are presented in a form of a plot in Fig. 1.2.1. We also present solutions of the Hamiltonian 12, given in the equation 17 (Figs. 2 and 3).

Particle confined to the plane in the uniform perpendicular magnetic field has been discussed in the absence of an electric field. However when electric field is present, one particle IQH wave function is simply shifted.[4]

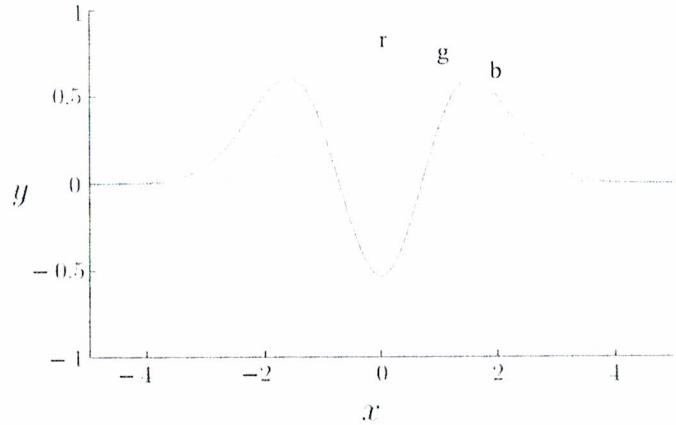

Fig. 1. Three leading Hermite functions normalized to one in L^2 norm. $H_0(x) \propto \exp(-x^2/2)$ – red curve, $H_1(x) \propto 2x\exp(-x^2/2)$ – blue curve and $H_2(x) \propto (4x^2 - 2)\exp(-x^2/2)$ – green curve.

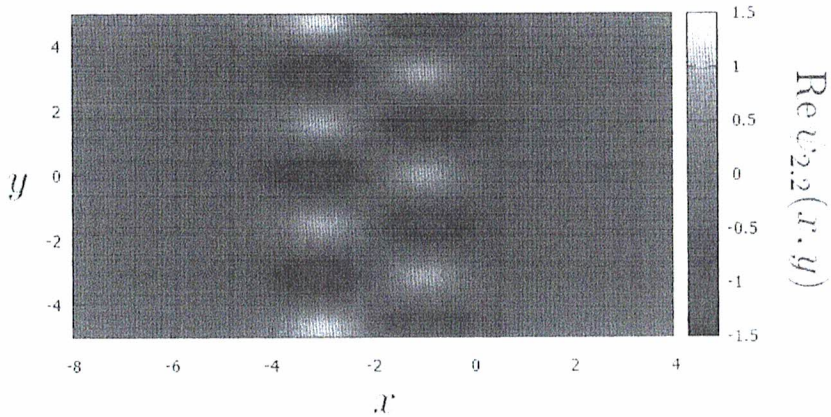

Fig. 2. Illustration of unnormalized solution $\psi_{2,2}(x, y)$ to the eigenvalue problem 12, given in 17 – real part of $\psi_{2,2}(x, y)$ (for magnetic length $\ell_B = 1$).

1.2.2. *One particle model: Symmetric gauge*

In a symmetric gauge magnetic vector potential equals

$$\mathbf{A} = \frac{B}{2}(-\mathbf{y}, \mathbf{x}, 0). \tag{19}$$

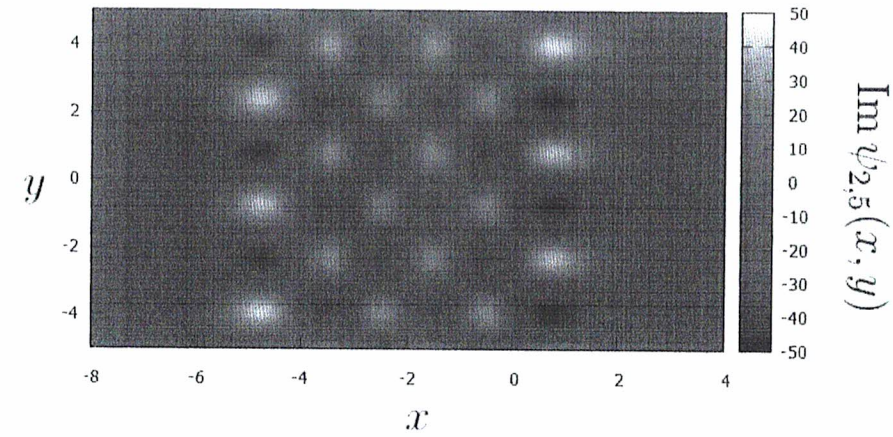

Fig. 3. Illustration of unnormalized solution $\psi_{2,5}(x, y)$ to the eigenvalue problem 12, given in 17 – imaginary part of $\psi_{2,5}$ (for magnetic length $\ell_B = 1$).

Then Hamiltonian may be written with usage of a raising and lowering operators[3,5,6]

$$H = \hbar\omega_c \left(a^\dagger a + \frac{1}{2}\right). \tag{20}$$

for

$$a = \frac{1}{\sqrt{2}}\left(\frac{z}{2\ell_B} + 2\ell_B\partial_z\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{z}{2\ell_B} - 2\ell_B\partial_z\right). \tag{21}$$

In an equation 21 we introduced variable $z \in \mathbb{C}$. It is defined as $z = x - iy$, $z = x + iy$. Unconventional definition of z is motivated by a convenient form of wave functions in the lowest Landau level (LLL).[3] Similarly one introduces operators $b, b^\dagger$

$$b = \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2\ell_B} + 2\ell_B\partial_{\bar{z}}\right), \quad b^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\bar{z}}{2\ell_B} - 2\ell_B\partial_{\bar{z}}\right). \tag{22}$$

Then

$$[a, a^\dagger] = [b, b^\dagger] = 1. \tag{23}$$

One defines z component of the angular momentum denoted L_Z (in XY-plane)

$$L_Z = -\hbar(b^\dagger b - a^\dagger a) \tag{24}$$

One notices

$$[H, L_Z] = 0. \tag{25}$$

thus it is natural to look for the eigenvectors of H in the set of the eigenvectors of L_Z. The eigenvalues of L_Z are given by $m\hbar$ for $m \in \mathbb{Z}, -n \leq m$ where n is a number of Landau level.

One notices solutions of the eigenvector problem are indexed by two quantum numbers n, m

$$|n, m\rangle = \frac{\left(b^\dagger\right)^{m+n}}{\sqrt{(m+n)!}} \frac{\left(a^\dagger\right)^n}{\sqrt{n!}} |0, 0\rangle. \tag{26}$$

where state $|0, 0\rangle$ corresponds to the Gaussian wave function

$$\Psi_{0,0} = \frac{1}{\sqrt{2\pi}} e^{-|z|^2/4}. \tag{27}$$

One notices

$$a|n, m\rangle = \sqrt{n}|n-1, m\rangle, \quad a^\dagger|n, m\rangle = \sqrt{n+1}|n+1, m\rangle. \tag{28}$$

and

$$b|n, m\rangle = \sqrt{m}|n, m-1\rangle. \quad b^\dagger|n, m\rangle = \sqrt{m+1}|n, m+1\rangle. \tag{29}$$

Moreover

$$a|0, 0\rangle = b|0, 0\rangle = 0. \tag{30}$$

Action of operators H and L_Z on eigenvectors is given by

$$H|n, m\rangle = \left(n + \frac{1}{2}\right)|n, m\rangle. \tag{31}$$

$$L_Z|n, m\rangle = \hbar(n - m)|n, m\rangle. \tag{32}$$

Direct calculations shows that in the LLL ($n = 0$), states $|0, m\rangle$ correspond to the products of radial Gaussian and certain power of z.

1.3. *Fractional quantum Hall effect*

The fractional quantum Hall effect (FQHE) was observed for the first time in 1982. Phenomenon occurs in temperatures close to the absolute zero in a presence of a strong magnetic field. Then collective states of quasi two-dimensional electrons reveal plateaus in the Hall resistivity at the filling factors given by rational numbers $\nu = \frac{p}{q} \in \mathbb{Q}$. The source of the FQHE is

distinct from its integer counterpart. It does not occur in the one particle model and electron-electron interaction plays significant role in it.

For filling factors ν less than one, it can be assumed all of the electrons lie within the LLL. Then kinetic energies of all electrons equals and the total kinetic energy can be subtracted from Hamiltonian. Then Hamiltonian for such system of N particles contains only part responsible for Coulomb interaction between electrons and can be written as

$$H = \sum_{1 \leq i < j \leq N} |\mathbf{r}_i - \mathbf{r}_j|^{-1}. \tag{33}$$

Where summation indexes runs over all particles and $\mathbf{r}_i$ is a position of the i-th particle. Since interaction energy cannot be treated as a small perturbation with respect to the kinetic energy standard, perturbation approach cannot be applied.

1.3.1. *Trial wave functions*

There is no general method of solving the FQH Hamiltonian (equation 33). Since neither exact nor based on perturbation method solutions can be constructed, phenomenological method, based on trial wave functions were developed. In 1983 Robert Laughlin proposed wave function describing ground state in LLL at filling factor $\nu = 1/q$ (q- odd).

Laughlin postulated ground state wave function at filling factor $\nu = 1/q$ should satisfy following conditions.[7]

1. The wave function for the whole system is an antisymmetrised product of single electron wave functions in the LLL.
2. Since the ground state is an eigenstate of the angular momentum operator, the wave function should be a homogeneous polynomial.
3. In order to keep electrons apart the wave function should be multiplied by a Jastrow type factor

$$\prod_{i<j} f(z_i - z_j) \tag{34}$$

where f is a homogeneous polynomial.

Then Laughlin gave an example of such function, now known as the Laughlin wave function.

$$\Phi_L^{1/m} = \prod (z_i - z_j)^m \exp\left(\frac{\sum_i |z_i|^2}{4\ell_*^2}\right). \tag{35}$$

The Laughlin wave function is not an exact ground sate of coulomb repulsion Hamiltonian, however theirs overlaps are extremely close to one and it is believed to capture all of physical properties of the phenomenon.

Since the Gaussian factor $\exp\left(\frac{\sum_i |z_i|^2}{4l_B^2}\right)$ appears in all LLL wave functions it is usually skipped. This allow us to focus on the polynomial part of wave functions, in particular we refer to $\prod_{i<j}(z_i - z_j)^m$ as the Laughlin wave function.

It is postulated that system created from Laughlin ground state by adiabatically inserting a magnetic flux (quasihole state) can be described by a function

$$\Phi_L^{1/m,\mathrm{QH}(z_0)} = \prod_i (z_i - z_0) \prod_{i<j} (z_i - z_j)^m. \tag{36}$$

Where quasihole is located at position z_0. Such quasihole is a quantized vortex, since the wave function changes by a phase of 2π when each electron moves around z_0. The creation operator of quasihole (in a center of the system) is given by

$$\prod_i z_i. \tag{37}$$

By conjugation one obtains analogical operator for quasiparticle

$$\prod_i -\partial_{z_i}. \tag{38}$$

When shifted, operators create quasihole and quasiparticle anywhere.

The Laughlin wave function is not the only successful example of trial wave function approach in FQHE. One can discuss the so called Moore-Read (MR) state wave function, sometimes refereed to as the Pfaffian state. The MR state is believed to give approximate description of FQH states in the second LL(for fermions $\nu = 1/2$ in the second LL and for bosons $\nu = 1$ in second LL). MR wave function is well defined for an even number of particles and is given by

$$\Psi_{MR}^m = \mathrm{pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j}^{N} (z_i - z_j)^{m+1}, \tag{39}$$

where $m = 0$ corresponds to the bosonic MR state and $m = 1$ to the fermionic one. Pfaffian $\mathrm{pf}(\cdot)$ is a function of antisymmetric matrix M of $2N \times 2N$ dimension

$$\mathrm{pf}(M_{ij}) = \sqrt{\det(M)}, \tag{40}$$

or alternatively

$$\mathrm{pf}(M) = \frac{1}{2^N N!} \sum_{\sigma \in S_{2N}} \mathrm{sgn}(\sigma) \prod_{i=1}^{N} M_{\sigma(2i-1),\sigma(2i)}. \tag{41}$$

The bosonic Moore-Read state is a densest zero energy ground state of the non physical 3-body interaction projection Hamiltonian H_3^2

$$H_3^2 = \sum_{i<j<k} P_3^2(i,j,k). \tag{42}$$

Where $P_3^2(i,j,k)$ is a projection operator onto subspace of the relative angular momentum two for each triplet of particles[8]

Other trial wave functions used in FQHE includes functions like the Read-Rezai (RR) associated with $\nu = 3/5$ state; the Gaffnian state (for bosons $\nu = 2/3$ and $\nu = 2/5$ for fermions) or the Haffnian state ($\nu = 1/3$ for fermions).

1.4. *Composite fermions*

The composite fermion theory, proposed by Jainendra Jain[1,3,6] explains appearance of many FQH states. Theory postulates existence of the composite fermions – quasiparticles, states of electrons and even number of flux quanta. For a system of N electrons in the perpendicular, uniform magnetic field B when each of them bounds $2p$ fluxes, in the limit of infinite N, each of the electrons experience an effective magnetic field B^*

$$B^* = B - 2pn\phi_0. \tag{43}$$

For a coefficient ϕ – number of flux quanta through the sample. Define $N_\phi = \frac{\phi}{\phi_0}$. Then the filling factor may be expressed as $\nu = N/N_\phi$. One gives filing factor of the composite fermions ν^* in terms of filling factor for the electrons ν

$$(\nu^*)^{-1} = \nu^{-1} - 2p. \tag{44}$$

Moreover

$$\nu = \frac{\nu^*}{2p\nu^* + 1}. \tag{45}$$

Construction of composite fermions gives form of the trial wave function of electrons Ψ^{FQHE} at filling factor ν in terms of wave function of composite fermions Ψ^{IQHE} at filling factor ν^*

$$\Psi^{FQHE} = P_{LLL} \Psi^{IQHE} \prod (z_i - z_j)^{2p}. \tag{46}$$

Where P_{LLL} is a projection into the LLL. The Jastrow factor $\prod_{i<j}(z_i - z_j)^{2p}$ assures each electron captures $2p$ fluxes.

Example of such construction is Laughlin state $\nu = \frac{1}{3}$ which corresponds to a completely filled Landau level of composite fermions.

1.4.1. *Haldane sphere*

Sometimes it is fruitful to examine FQHE when a system of electrons is projected from the plane on the so called Haldane sphere.[1,3,6,9,10] The Haldane sphere is the two dimensional sphere containing electrons. Magnetic field, perpendicular to its surface is provided by a Dirac monopole in a center of the sphere. Notice a magnetic flux through the surface of considered sphere, is quantized to $2Q\phi_0$, where $2Q$ is integer. Spherical geometry resolves problem of the boundary and as such can be used in the study of bulk properties. Moreover Landau levels on the sphere are only finitely degenerate. Such model of FQHE is typically used in numerical study of FQHE (direct diagonalisation of Hamiltonian). Value of the radial magnetic field B on the surface of the sphere of radius r is given by

$$B = \frac{2Q\phi_0}{4\pi r^2}. \tag{47}$$

One introduces coordinates u, v on the Haldane sphere. u, v are more useful in the context of FQHE than standard θ and φ. New coordinates can be expressed in terms of standard ones

$$u = \cos(\theta/2)e^{i\varphi/2}, \quad v = \sin(\theta/2)e^{-i\varphi/2}. \tag{48}$$

Consider a Hilbert space of square integrable functions over $\mathbb{C}$ defined on a sphere, with respect to the standard, rotation invariant measure $d\Omega = \sin\theta d\theta d\varphi$. Let H_S denote subspace of Hilbert space, spanned by homogeneous polynomials in u, v.[10] Set of functions $\{c_{Q,m}\}_{Q,m}$ provide an orthonormal basis in H_S

$$c_{Q,m} = \sqrt{\frac{2Q+1}{4\pi}\binom{2Q}{Q+m}}u^{Q+m}v^{Q-m}, \tag{49}$$

where $m \in \{-Q, -Q+1, \ldots, Q\}$.

Transition from the space H_S to the space of wave functions in the LLL is established by a function Γ

$$\Gamma\left(\sum^{2Q} c_i u^{2Q-i}v^i\right) = \sum^{2Q} c_i z^k, \tag{50}$$

where $c_i \in \mathbb{C}$ are coefficients.

Γ is multiplicative and one can extend this mapping onto many particle wave functions.

We give an example how function Γ acts on polynomials by mapping the Jastrow factor from the plane

$$\Gamma^{-1}\left(\prod_{i<j}(z_i - z_j)^m\right) = \prod_{i<j}(u_i v_j - v_j u_i)^m. \tag{51}$$

One writes angular momentum operators on the Haldane sphere

$$L^X = \frac{1}{2}\left(v\partial_u + u\partial_v\right), \tag{52a}$$

$$L^Y = \frac{i}{2}\left(v\partial_u - u\partial_v\right), \tag{52b}$$

$$L^Z = \frac{1}{2}\left(u\partial_u - v\partial_v\right). \tag{52c}$$

There are standard raising and lowering operators

$$L^+ = L^X + iL^Y = -u\partial_v, \tag{53a}$$

$$L^- = L^X - iL^Y = -v\partial_u. \tag{53b}$$

2. Theory of symmetric functions

Theory of symmetric functions is a field of mathematics with many applications. Objects developed in this theory appear in physics and various branches of mathematics. Symmetric polynomials occur in the Galois theory, specific symmetric functions naturally arise in the asymptotic group representation theory, combinatorics and algebraic combinatorics. Schur polynomials are characters of irreducible representations of the general linear groups. The main object of our interest in this paper – Jack polynomials are one parameter deformations of Schur polynomials. Moreover Jack polynomials occur in the study of free probability and in physics of many body systems.

We give a brief introduction to the symmetric functions theory. We define and discus fundamental objects and concepts of this theory, starting with partitions. Then we give construction of the ring of symmetric polynomials and define important families of symmetric and antisymmetric polynomials.

2.1. *Partitions*

The partition λ[11,12] is a sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots \lambda_j, \ldots). \tag{54}$$

of non negative integers in non increasing order

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j \geq \ldots. \tag{55}$$

Partition can be an infinite sequence, however only finitely many elements can be nonzero. Usually when partition indexes polynomial, sequence length corresponds to the number of variables of polynomial. The non zero λ_i are called the *parts* of λ, the number of parts is the *length* of λ and it is denoted by $\ell(\lambda)$. The sum of the parts of λ is called the *weight* and is denoted $|\lambda|$

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_j + \ldots. \tag{56}$$

$m(\lambda, i)$ is a number of parts of λ equal i. Sometimes one does not distinguish between two sequences varying only by a string of zeros at the end.[11] In such situation one regard $(3,3,1)$ and $(3,3,1,0)$ as the same partition.

To keep notation short one can represent partition in its *frequency representation*. Frequency representation indicates how many times given number occurs in partition

$$\lambda = \left(1^{m(\lambda,1)}\, 2^{m(\lambda,2)} \ \ldots\ i^{m(\lambda,i)} \ \ldots\right). \tag{57}$$

Information about number of occurrences of parts of partition is very useful in physics of fractional quantum Hall effect. However due to the small indexes of frequency representation, it is more practical to use the *occupation number configuration*[20]

$$\lambda = [m(\lambda,0)\ m(\lambda,1)\ m(\lambda,2)\ \ldots]. \tag{58}$$

For example partition $(3,3,1)$ in the standard notation, is equivalent to $(1^1 2^0 3^2)$ in the frequency representation and to $[0\ 1\ 0\ 2]$ in the occupational representation.

For two partitions there is a natural operation of addition

$$\begin{aligned}\lambda + \mu &= (\lambda_1, \lambda_2, \ldots \lambda_j, \ldots) + (\mu_1, \mu_2, \ldots \mu_j, \ldots) \\ &= (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots \lambda_j + \mu_2, \ldots).\end{aligned} \tag{59}$$

The *natural order* of partitions is defined as follows

$$\lambda > \mu \Leftrightarrow \forall i \cdot \lambda_1 + \lambda_2 + \ldots + \lambda_i > \mu_1 + \mu_2 + \ldots + \mu_i \tag{60}$$

Then one says λ *dominates* μ. For example $(3,3,1) \geq (3,2,2)$. The natural order is not a total order and incomparable partitions exists. For example $(3,1,1,1) \not\leq (2,2,2,0)$ because for the first parts $3 > 2$ and $(3,1,1,1) \not\geq (2,2,2,0)$ since the sums of first three parts give $5 < 6$. Total order consistent with the natural order is the *reverse lexicographic order* denoted with the symbol "$\overset{R}{\geq}$".[14] One writes $\lambda \overset{R}{\geq} \mu$ when either $\lambda = \mu$ or the first non vanishing difference $\lambda_i - \mu_i$ is positive. Considering the previous example $(3,1,1,1) \overset{R}{\geq} (2,2,2,0)$.

2.1.1. *Diagrams*

Partition may be represented in graphical form as a *Young diagram* (Young tableau). A Young diagram of a partition λ is a set of the $|\lambda|$ boxes. Boxes are arranged in rows and columns in a way that indicating consecutive parts of the partition. In the so called English notation, there are λ_i left aligned boxes in i-th row starting from the top. In this notation number of rows equals $\ell(\lambda)$ (see Fig. 4).

Formally one can define a Young diagram as a subset of a plane composed out of 1×1 boxes. Center of each box is placed in an integer point $\langle i, j\rangle \in \mathbb{Z}^2$ and first box (right-top) in a point $\langle 0,0\rangle$. The empty set is also Young diagram, it represents the partition of weight 0.

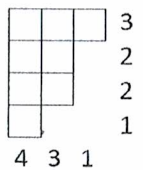

Fig. 4. Young diagram in English notation of a partition $(3,2,2,1)$ and numbers of the boxes in each row and column.

For a given partition λ one defines a conjugated partition λ' by giving its parts. i-th part of the λ' equals to the quantity of boxes in i-th column of the Young diagram of λ. As such the Young diagram of the λ' can be obtained from the diagram of λ by taking mirror reflection with respect to the diagonal line passing through points $\{\langle i, i\rangle\}_{i\in\mathbb{Z}}$. For example $(3,2,2,1)' = (4,3,1)$ (see Fig. 4)

Conjugated partition can be defined without referring to Young diagrams. Parts of conjugated partition are given by a formula

$$\lambda'_i = |\{j : \lambda_j \geq i\}|. \tag{61}$$

Simple properties of the conjugation may be derived: $(\lambda')' = \lambda$. $\lambda'_1 = \ell(\lambda)$, $\lambda_1 = \ell(\lambda')$ and $m(\lambda, i) = \lambda'_i - \lambda'_{i+1}$. Moreover conjugation reverses the natural order

$$\lambda \geq \mu \Leftrightarrow \lambda' \leq \mu'. \tag{62}$$

Denote

$$\delta_N = (N-1, N-2, \ldots, 1, 0). \tag{63}$$

When the length of δ is clear we skip the index.

Conciser a vector of integers $a = (a_1, a_2, \ldots, a_N) \in \mathbb{Z}^N$. There is at least one permutation of vector components that rearranges those components into non increasing order. Such rearranged vector is unique and it is denoted a^*.

For any natural numbers $1 \leq i < j \leq N$ and ℓ the *raising operator* $R^\ell_{i,j} : \mathbb{Z}^N \to \mathbb{Z}^N$ acts on a vector $a \in \mathbb{Z}^N$ as follows

$$R^\ell_{i,j}(a_1, a_2, \ldots, a_i, \ldots, a_j, \ldots a_N) = (a_1, a_2, \ldots, a_i - \ell, \ldots, a_j + \ell, \ldots a_N). \tag{64}$$

2.2. *The ring of symmetric polynomials*

The polynomial $P(x_1, x_2, \ldots, x_N)$ in N variables is symmetric if it is invariant under all of the permutations of variables i.e. for any permutation $\sigma \in S_N$

$$P(x_1, x_2, \ldots, x_N) = P(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(N)}). \tag{65}$$

Similarly the polynomial $A(x_1, x_2, \ldots, x_N)$ in N variables is antisymmetric if for any permutation $\sigma \in S_N$

$$A(x_1, x_2, \ldots, x_N) = \operatorname{sgn}(\sigma) A(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(N)}). \tag{66}$$

Let Λ^k_N denote the additive group of symmetric, homogeneous polynomials in N variables and degree k, with zero polynomial (by the convention, 0 is homogeneous polynomial of every degree). Then Λ_N equipped with standard addition and multiplication of polynomials is a graded ring of symmetric polynomials in N variables[11]

$$\Lambda_N = \bigoplus \Lambda^k_N. \tag{67}$$

Presented construction can be performed for polynomials over various fields e.g. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$. Usually the choice of field is not significant.

Analogical construction can be performed for for infinitely many variables with an usage of an inverse limit. Ring of the symmetric functions in infinitely many variables is denoted Λ. Formally elements of Λ (unlike those of Λ_N) are no longer polynomials but series of polynomials and the term "function" is being used instead of polynomial.

2.3. *Monomial symmetric functions m_λ*

One defines the *monomial symmetric functions* - m_λ as

$$m_\lambda(x_1, x_2, \ldots, x_N) = \sum_{\text{distinct permutations } \alpha \text{ of } \lambda} x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \ldots \cdot x_N^{\alpha_N}. \tag{68}$$

or equivalently

$$m_\lambda(x_1, x_2, \ldots, x_N) = F(\lambda) \cdot \sum_{\sigma \in S_n} x_1^{\lambda_{\sigma(1)}} \cdot x_2^{\lambda_{\sigma(2)}} \cdot \ldots \cdot x_N^{\lambda_{\sigma(N)}}. \tag{69}$$

For $F(\lambda)$ the normalizing factor, given by

$$F(\lambda) = \frac{1}{m(\lambda, 0)! \cdot m(\lambda, 1)! \cdot \ldots}. \tag{70}$$

For example

$$m_{(3,1,1)}(x, y, z) = x^3yz + xy^3z + xyz^3 = (x^2 + y^2 + z^2)xyz,$$

$$m_{(2,1)}(x_1, \ldots, x_N) = \sum_{i \neq j} x_i^2 x_j.$$

The monomial symmetric functions are the $\mathbb{Z}$-basis of Λ_N.[11]

2.4. *Schur polynomials*

2.4.1. *Vandermonde determinant*

A square Vandermonde matrix in variables $x_1, x_2, \ldots, x_N$, is a matrix with the terms of a geometric progression in each row

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \ldots & x_N^{N-1} \\ 1 & x_2 & x_2^2 & \ldots & x_N^{N-1} \\ 1 & x_3 & x_3^2 & \ldots & x_N^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & & x_N^{N-1} \end{bmatrix}. \tag{71}$$

Or shorter $V_{ij} = x_i^{j-1}$. The determinant of this matrix is denoted $D(x_1, \ldots, x_N)$ or simply D when variables are known. Vandermonde determinant is an object of great importance in linear algebra and the theory of symmetric functions. It can be represented as

$$D(x_1, \ldots, x_N) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \cdot x_{\sigma(1)}^{N-1} \cdot x_{\sigma(2)}^{N-2} \cdot \ldots \cdot x_{\sigma(N-1)}^{1} \cdot x_{\sigma(N)}^{0} = \prod_{1 \le i < j \le N} (x_i - x_j). \tag{72}$$

Every antisymmetric polynomial is N variables is divisible by the Vandermonde determinant. $D(x_1, \ldots, x_N)$ has a degree $\binom{N}{2}$.

2.4.2. *Slater determinants*

Slater determinants sl_ν plays similar role in the ring of antisymmetric functions as monomials in the ring of symmetric functions. Slater determinants sl_ν (sometimes called antisymmetric polynomials a_ν) are indexed by a partition ν and given by a following formula

$$\mathrm{sl}_\nu(x_1, x_2, \ldots, x_N) = \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) \cdot x_{\sigma(1)}^{\nu_1} \cdot x_{\sigma(2)}^{\nu_2} \cdot \ldots \cdot x_{\sigma(N)}^{\nu_N}. \tag{73}$$

Assume the partition ν has two parts equal, then the transposition interchanging those two parts would not affect it. Since transpositions are odd permutations, polynomial sl_ν would equal $-\mathrm{sl}_\nu$ therefore $\mathrm{sl}_\nu = 0$. Thus in order for sl_ν to be nonzero, no part of ν can occur more than once. Let δ denote the partition from formula 63: $\delta = (N-1, \ldots, 0)$. Then a partition ν with no repeated parts can be represented as $\nu = \delta + \lambda$ for some partition λ.

Slater determinants are $\mathbb{Z}$-basis of A_N, where A_N is a space of antisymmetric polynomials of N variables.[11] One can notice sl_δ is a Vandermonde determinant. For example

$$\mathrm{sl}(x, y, z)_\delta = \mathrm{sl}(x, y, z)_{(2,1,0)} = D(x, y, z) = x^2 y - x^2 z + y^2 z - y^2 x + z^2 x - z^2 y.$$

2.4.3. *Definition of Schur polynomials*

The Schur polynomials s_λ are important family of polynomials in the ring of the symmetric polynomials, they are defined as a quotient of antisymmetric polynomials

$$s_\lambda = \frac{\mathrm{sl}_{\lambda+\delta}}{\mathrm{sl}_\delta}. \tag{74}$$

The multiplication by $D = \mathrm{sl}_\delta$ is an isomorphism of Λ_N onto A_N. Since $\{s_\lambda\}_\lambda$ are transformed in basis $\{\mathrm{sl}_{\lambda+\delta}\}_\lambda$ one concludes Schur polynomials form $\mathbb{Z}$-basis of Λ_N.[11] Schur polynomial s_λ is homogeneous symmetric polynomial of a degree $|\lambda|$. For example

$$s_{(2,1,1)}(x, y, z) = \frac{a_{(4,2,1)}}{a_{(2,1,0)}} = xyz(x + y + z).$$

Representation of Schur polynomials in monomial basis is triangular i.e. satisfy

$$s_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\lambda\mu} m_\mu. \tag{75}$$

for the coefficients $K_{\lambda\mu} \ge 0$ — Kostka numbers.[13]

3. Jack polynomials

For a real number $\alpha \in \mathbb{R}$, consider a field $\mathbb{Q}(\alpha)$ of rational functions in α

$$\mathbb{Q}(\alpha) = \left\{ \frac{p(\alpha)}{q(\alpha)} \in \mathbb{R} : \; q(\alpha) \neq 0; \; p(\alpha), q(\alpha) \text{ - polynomials over rationals} \right\}. \tag{76}$$

$\mathbb{Q}(\alpha)$ is the smallest (in a sense of inclusion) field containing rational numbers and α. For $\alpha \in \mathbb{Q}$ field reduces to the rational numbers $\mathbb{Q}(\alpha) = \mathbb{Q}$. Let $\Lambda_N \otimes \mathbb{Q}(\alpha)$ denote the ring of all symmetric polynomials in N variables over the field $\mathbb{Q}(\alpha)$.[14]

The Hamiltonian in the Calogero-Sutherland-Moser model[15] (also called Hamiltonian of the Calogero-Sutherland model[16] or Laplace-Beltrami operator[14]) is defined as follows $H^0(\alpha) : \Lambda_N \otimes \mathbb{Q}(\alpha) \to \Lambda_N \otimes \mathbb{Q}(\alpha)$:

$$H^{CSM}(\alpha) = \alpha \sum_{i=1}^{N} (x_i \partial_i)(x_i \partial_i) + \sum_{1 \le i < j \le N} \left(\frac{x_i + x_j}{x_i - x_j} \right) (x_i \partial_i - x_j \partial_j). \tag{77}$$

For a real number α and a partition λ a Jack polynomial J_λ^α is a polynomial over $\mathbb{Q}(\alpha)$.[17] It is defined (up to normalization) as an eigenvector of $H^{CSM}(\alpha)$

$$H^{CSM} J_\lambda^\alpha = d_\lambda(\alpha) \cdot J_\lambda^\alpha, \tag{78}$$

where eigenvalue $d_\lambda(\alpha)$ equals

$$d_\lambda(\alpha) = \sum_{i=1}^{N} \left(\alpha \lambda_i^2 + (N + 1 - 2i) \lambda_i \right). \tag{79}$$

When parameter α is known one skips it and writes simply H^{CSM} and d_λ

Analogically to the Schur polynomials The Jack polynomials (Jacks) are triangular in monomial basis i.e.

$$J^{\alpha}_{\lambda} = m_{\lambda} + \sum_{\mu<\lambda} v_{\lambda\mu}(\alpha) m_{\mu}. \tag{80}$$

For $\alpha > 0$ coefficients $v_{\lambda\mu}(\alpha)$ are positive. Moreover coefficients can be expressed as an inverse of polynomials in α with no nonnegative root.

Remark 3.1. Form of expansion in equation 80 fixes problem of normalization of Jack polynomials.

Remark 3.2. Jack polynomials form a complete set of the eigenvectors of H_0.

Few examples of Jack polynomials

$$J^{\alpha}_{(1,1)} = m_{(1,1)}.$$

$$J^{\alpha}_{(2,1)} = m_{(2,1)} + \frac{6}{2+\alpha} m_{(1,1,1)}.$$

$$J^{\alpha}_{(3)} = m_{(3)} + \frac{3}{2+\alpha} m_{(2,1)} + \frac{6}{(1+\alpha)(2+\alpha)} m_{(1,1,1)}.$$

$$J^{1}_{\lambda} = s_{\lambda}, \quad J^{\infty}_{\lambda} = m_{\lambda}.$$

Jack polynomials are generalization of many types of the symmetric polynomials including: Schur polynomials $\alpha = 1$, zonal polynomials $\alpha = 2$, quaternion zonal polynomial $\alpha = \frac{1}{2}$, monomial symmetric functions $\alpha = \infty$ and elementary symmetric functions $\alpha = 0$. Jack polynomials are special, limit case of Macdonald polynomials $P^{q,t}_{\lambda}$ for $q = t^{\alpha}$, $t \to 1$.[11–13]

In order to construct Jack polynomials in monomial basis we use two following theorems.

Theorem 3.1. ***Action of H^{CSM} on the monomial symmetric functions (K. Sogo)***
Action of H^{CSM} on a monomial function is given by

$$H^{CSM}(m_{\lambda}) = d_{\lambda} m_{\lambda} + \sum_{\mu<\lambda} C'_{\lambda\mu}\, m_{\mu}. \tag{81}$$

For eigenvalues d_{λ} defined in (79), coefficients $C'_{\lambda\mu} \in \mathbb{N}$ are nonzero only when there exist some i, j, ℓ such $i < j$, $1 < \ell < \left|\frac{\lambda_i - \lambda_j}{}\right|$ and $(R^{\ell}{}_{ij}(\lambda))^{*} =$

μ, then

$$C'_{\lambda\mu} = \begin{cases} (\lambda_i - \lambda_j) m(\mu, \lambda_i - \ell)(m(\mu, \lambda_i - \ell) - 1) & \text{if} \quad \lambda_i - \ell = \lambda_j + \ell \\ (\lambda_i - \lambda_j) 2 m(\mu, \lambda_i - \ell) m(\mu, \lambda_j + \ell) & \text{if} \quad \lambda_i - \ell \neq \lambda_j + \ell \end{cases}. \tag{82}$$

Proof of the theorem 3.1 can be found in Ref. 15.

Remark 3.3. Theorem 3.1 reveals that action of an operator H^{CSM} on the monomials is triangular i.e. a result of $H^{CSM} m_{\lambda}$ is a sum of the monomials indexed by the partitions dominated by λ. This property is analogical to an action of the triangular matrix. Then coefficients d_{λ} are diagonal elements of the operator H^{CSM}.

Theorem 3.2. ***Determinantal expression for Jack Polynomials (K. Sogo, L. Lapointe, A. Lascoux, J. Morse)***
Let $\mu^{(1)} \overset{R}{<} \mu^{(2)} \overset{R}{<} \ldots \overset{R}{<} \mu^{(n)} = \lambda$ be a reverse lexicographic ordering of all partitions dominated by λ. Then an eigenvector of H^{CSM} corresponding to λ Jack polynomial J^{α}_{λ} is proportional to the determinant of quasi-triangular matrix (one line below diagonal is nonzero).

$$J^{\alpha}_{\lambda} \propto \det \begin{bmatrix} m_{\mu^{(1)}} & m_{\mu^{(2)}} & \ldots\ldots & m_{\mu^{(n-1)}} & m_{\mu^{(n)}} \\ d_{\mu^{(1)}} - d_{\mu^{(n)}} & C'_{\mu^{(2)}\mu^{(1)}} & \ldots\ldots & C'_{\mu^{(n-1)}\mu^{(1)}} & C'_{\mu^{(n)}\mu^{(1)}} \\ 0 & d_{\mu^{(2)}} - d_{\mu^{(n)}} & \ldots\ldots & C'_{\mu^{(n-1)}\mu^{(2)}} & C'_{\mu^{(n)}\mu^{(2)}} \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots\ddots & \vdots & \vdots \\ 0 & 0 & \ldots\ 0 & d_{\mu^{(n-1)}} - d_{\mu^{(n)}} & C'_{\mu^{(n)}\mu^{(n-1)}} \end{bmatrix}. \tag{83}$$

The eigenvalues d_{λ} are defined in equation (79). The matrix is filled with monomials in the first row and numbers elsewhere. The elements below diagonal are differences of eigenvalues of H^{CSM} defined in formula 79. Rest of the elements are coefficients $C'_{\mu^{(i)}\mu^{(j)}}$ defined in the formula 82.

Proof of the theorem 3.2 can be found in Ref. 16.

Remark 3.4. When combined with known properties of quasi diagonal matrix[16] theorem 3.2 (Determinantal expression for Jack Polynomials) gives recursion formula for expansion of Jack polynomials in monomial basis

Rewrite equation 80:

$$J^{\alpha}_{\lambda} = m_{\lambda} + \sum_{\mu<\lambda} v_{\lambda\mu} m_{\mu}.$$

for $\mu^{(1)} \overset{R}{<} \mu^{(2)} \overset{R}{<} \ldots \overset{R}{<} \mu^{(n)} = \lambda$ a reverse lexicographic ordering of all partitions dominated by λ, one has to put $v_{\lambda\lambda}(\alpha) = v_{\mu^{(n)}\mu^{(n)}}(\alpha) = 1$ and obtain other coefficients from the recursion formula

$$v_{\lambda\mu^{(i)}} = \frac{1}{d_{\lambda} - d_{\mu^{(i)}}} \sum_{j=i+1}^{n} C^{\mu^{(j)}}_{\mu^{(i)}\mu^{(n)}} v_{\lambda\mu^{(j)}}. \tag{84}$$

3.1. *Fermionic Jack polynomials*

The Jack fermionic polynomials S^{α}_{μ}[18,19] are antisymmetric analogue of Jack symmetric polynomials. They are defined as a product of Jack and Vandermonde determinant

$$S^{\alpha}_{\lambda+\delta}(x_1, \ldots, x_N) = J^{\alpha}_{\lambda}(x_1, \ldots, x_N) D(x_1, \ldots, x_N). \tag{85}$$

Slater determinants $\mathrm{sl}_{\lambda+\delta}$ are special case of fermionic Jack polynomials ($\alpha = 1$). Therefore $S^{\alpha}_{\lambda+\delta}$ may be treated as a continuous deformation of antisymmetric polynomials. Obviously fermionic Jack polynomial $S^{\alpha}_{\lambda+\delta}$ is well defined as long as Jack polynomial J^{α}_{λ} is well defined. In particularity fermionic Jack polynomial has only finitely many negative poles. One can expand $S^{\alpha}_{\lambda+\delta}$ in Slater determinant basis

$$S^{\alpha}_{\lambda+\delta} = \mathrm{sl}_{\lambda+\delta} + \sum_{\mu<\lambda} b_{\lambda\mu}(\alpha) \mathrm{sl}_{\mu+\delta}. \tag{86}$$

for $b_{\lambda\mu}(\alpha) \in \mathbb{Q}(\alpha)$. When α is clear we write simply $b_{\lambda\mu}$.

Fermionic Jack polynomials are eigenvectors of fermionic Laplace-Beltrami operator[18,19] $H^{F}_{LB}(\alpha)$

$$H^{F}_{LB}(\alpha) = \sum_{i=1}^{N} (x_i\partial_i)(x_i\partial_i) + \left(\frac{1}{\alpha} - 1\right) \sum_{1 \le i < j \le N} \frac{x_i + x_j}{x_i - x_j}(x_i\partial_i - x_i\partial_j) - 2\frac{{x_i}^2 + {x_j}^2}{(x_i - x_j)^2}. \tag{87}$$

$H^{F}_{LB}(\alpha) S_{\lambda+\delta} = E_{\lambda}(\alpha) S_{\lambda+\delta}$, where the eigenvalues are given by

$$E_{\lambda}(\alpha) = \sum_{i=1}^{N} \lambda_i \left(\lambda_i - 2\left(\frac{1}{\alpha} - 1\right) i\right) + \left(\frac{1}{\alpha} - 1\right)\left((N+1)\sum_{i=1}^{N} \lambda_i - N(N-1)\right). \tag{88}$$

When parameter α is clear we skip it and write simply $H^{F} = E_{\lambda}$

Recursion formula for fermionic Jacks in terms of antisymmetric polynomials basis had been derived by Bernevig and Regnault.[18,19] Bernevig and Regnault used the same method of constructing determinetal and later recursion form of eigenvectors of certain operators as presented in subsections above. Recursion formula holds. For $\mu^{(1)} \overset{R}{<} \mu^{(2)} \overset{R}{<} \ldots \overset{R}{<} \mu^{(n)} = \lambda$ a reverse lexicographic ordering of all partitions dominated by λ, one has to put $b_{\lambda\lambda} = b_{\mu^{(n)}\mu^{(n)}} = 1$ and obtain other coefficients from the recursion formula

$$b_{\lambda\mu^{(i)}} = \frac{2\left(\frac{1}{\alpha} - 1\right)}{E_{\lambda} - E_{\mu^{(i)}}} \sum_{j=i+1}^{n} C^{E}_{\mu^{(j)}\mu^{(n)}} b_{\lambda\mu^{(j)}}. \tag{89}$$

For coefficients $C^{E}_{\lambda\mu} \in \mathbb{Z}$ being nonzero only when there exist some i, j, t, such $i < j$, $1 \le t \le \left\lfloor \frac{\lambda_i - \lambda_j}{2} \right\rfloor$, $\left(R^{t}_{i,j}(\lambda)\right)^{T} = \mu$ and $\lambda_i - t \not/ \lambda_j + t$, then

$$C^{E}_{\lambda\mu} = (\lambda_i - \lambda_j - 2t)(-1)^{N_{SW}}. \tag{90}$$

Where N_{SW} is the number of the rising operators of a form $R^{1}_{i,i+1}$ or $R^{1}_{i,i-1}$ (swaps) needed, to obtain partition μ from λ.

3.2. *Jack polynomials indexed by negative α parameter*

Originally Jack polynomials were defied only for positive values of α[17] (definition involved a scalar product $\langle \cdot, \cdot \rangle_{\alpha}$ which had sense only for $\alpha > 0$.[11]) Nonetheless further development allowed for alternative definitions of Jack polynomials, including one given in this section. As one notices recursion construction of Jack polynomials show coefficients of Jacks in monomial basis as inverse polynomials in α that could be easily extended to negative α. Let us rewrite already given example of Jack polynomial

$$J^{\alpha}_{(3)} = v_{(3),(3)}(\alpha) m_{(3)} + v_{(3),(2,1)}(\alpha) m_{(2,1)} + v_{(3),(1,1,1)}(\alpha) m_{(1,1,1)}$$

$$= m_{(3)} + \frac{3}{2+\alpha} m_{(2,1)} + \frac{6}{(1+\alpha)(2+\alpha)} m_{(1,1,1)}.$$

Fig. 5 shows values of $v_{(3),(3)}(\alpha)$, $v_{(3),(2,1)}(\alpha)$, $v_{(3),(1,1,1)}(\alpha)$ coefficients as functions of α. Fig. 6 shows absolute values of the same coefficients normalized in a way that sum of theirs squares sum up to one.

One can notice (see Fig. 5) Jack polynomial $J^{\alpha}_{(3)}$ has two poles at $\alpha = -2, -1$. For $\alpha = -1$ coefficient $v_{\ldots,\ldots,\ldots}$ diverges to the infinity while

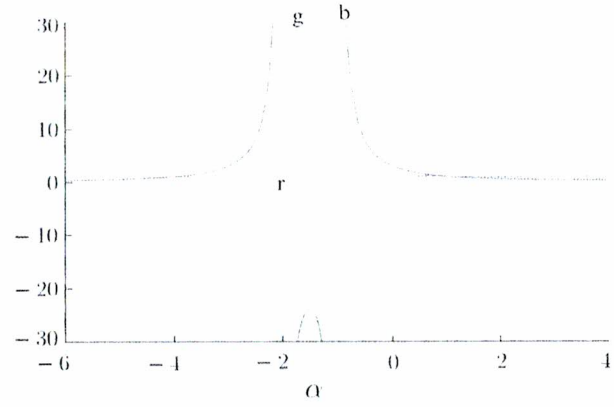

Fig. 5. Values of $v_{(3),(3)}(\alpha)$ (red), $v_{(3),(2,1)}(\alpha)$ (green), $v_{(3),(1,1,1)}(\alpha)$ (blue) coefficient as functions of α.

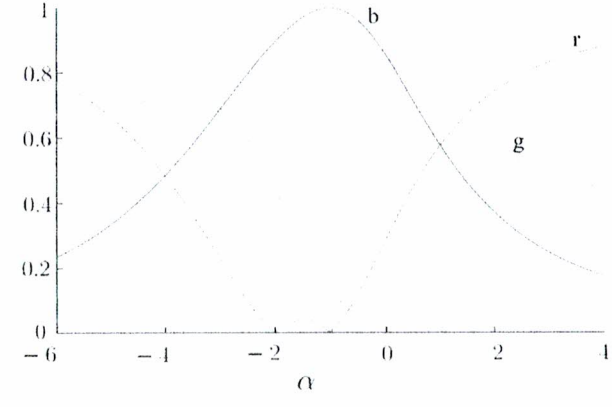

Fig. 6. Absolute values of $v_{(3),(3)}(\alpha)$ (red), $v_{(3),(2,1)}(\alpha)$ (green), $v_{(3),(1,1,1)}(\alpha)$ (blue) coefficient normalized to one in a square, as functions of α.

other remain finite. For $\alpha = -2$ coefficients $v_{(3),(1,1,1)}$ and $v_{(3),(2,1)}$ diverges while $v_{(3),(3)} = 1$. Moreover

$$\lim_{\alpha \to -2} \frac{v_{(3),(1,1,1)}(\alpha)}{v_{(3),(2,1)}(\alpha)} = -2. \tag{91}$$

Considerations of those limit justify definition of $J^{\alpha}_{(3)}$ at points $\alpha = -2, -1$ up to normalization as

$$J^{-2}_{(3)} \propto m_{(2,1)} - 2m_{(1,1,1)}, \quad J^{-1}_{(3)} \propto m_{(1,1,1)}.$$

Such approach gives definition of Jack polynomials even when α is its pole

$$J^{\alpha}_{\lambda} = \sum_{\mu \leq \lambda} w_{\lambda\mu} m_{\mu}. \tag{92}$$

where

$$w_{\lambda\mu} = \lim_{x \to \alpha} \frac{v_{\lambda\mu}(x)}{v_{\lambda\nu}(x)} \tag{93}$$

and $v_{\lambda\mu}$ is the coefficients that diverges in the highest power. Notice that for such definition first nonzero coefficient $w_{\lambda\mu}$ always equals one.

3.3. *Jack states of quantum Hall effect*

Now we a give brief overview of the Jack states FQH states related to Jack polynomials. It was pointed out[20 22] that analysis of the angular momentum operators on the sphere can be useful in determination of Jack states. By stereographic projection, operators can be applied to functions on the plane. Then operators take form

$$L^{+} = E_0. \tag{94a}$$

$$L^{-} = N_{\Phi} Z - E_2. \tag{94b}$$

$$L^{Z} = \frac{1}{2} N N_{\Phi} - E_1. \tag{94c}$$

where

$$E_n = \sum_{i=1}^{N} z_i^n \partial_i. \tag{95a}$$

$$Z = \sum_{i=1}^{N} z_i. \tag{95b}$$

and N_{Φ} is a strength of a magnetic monopole inside Haldane sphere. Papers 20 25 gave necessary conditions for both partition and real parameter of the Jack polynomial to be a candidate for FQH wave function. Bernevig and Haldane required a Jack wave function ψ, to be annihilated by raising and lowering operators on the sphere $L^{+}\psi = 0, L^{-}\psi = 0$ (highest weight HW and lowest weight LW conditions, respectively).[26] This analysis reveals the condition (necessary but not sufficient)

$$N - \ell(\lambda) + 1 + \alpha(\lambda_{\ell(\lambda)} - 1) = 0 \tag{96}$$

The equation 96 and further considerations, implies the real parameter equals $\alpha \equiv \alpha_{k,r} = -(k+1)/(r-1)$ for $(k+1)$ and $(r-1)$ both positive integers and coprime.[20 22] Partition indexing Jack polynomial is of a form

$$\lambda = [n_0, 0^{s(r-1)}, k, 0^{r-1}, k, 0^{r-1}, k, \ldots, k], \tag{97}$$

where 0^{r-1} means a sequence of $r-1$ zeros and $n_0 = (k+1)s - 1$. Such partition is denoted $\lambda^0_{k,r,s}$. The case $s = 1$ was recognized to provide many FQH ground states at filling factor $\nu = k/r$. Cases $s > 1$ are related to the quasiparticle states. Denote partition $\lambda^0_{k,r,s=1} = \lambda^0_{k,r}$.[20 22]

As we already pointed out Jack polynomials indexed by $\lambda^0_{k,r}$ and $\alpha_{k,r}$ are related to bosonic FQH states of filling factors $\nu = k/r$. Now we discuss few explicit Jack states. The bosonic Laughlin wave function for state $\nu = 1/r$ (r- even) can be represented as a product of Gaussian and symmetric Jack polynomial

$$\Phi_L^{1/r} = \prod_{i<j}^{N} (z_i - z_j)^r = J^{\alpha_{1,r}}_{\lambda^0(1,r)}. \tag{98}$$

As trivially follows fermionic Laughlin wave functions for a state $1/r$ also are Jack states for partition $\lambda^0(1,r)$ and real parameter $\alpha_{1,(r-1)}$.

The Moore-Read state (for fermions $\nu = 5/2,\ 1/2$ in the second LL and for bosons $\nu = 1$ in second LL) is also a Jack state

$$\Psi^0_{MR} = J^{\alpha_{2,2}}_{\lambda^0_{(2,2)}}; \quad \Psi^1_{MR} = S^{\alpha_{2,2}}_{\lambda^0_{(2,2)}+\delta}. \tag{99}$$

The Moore-Read state along the Laughlin $\nu = 1/3$ is a part of Read-Rezai parafermion sequence of states $\nu = k/(2 + kM)$. Whole series can be given as Jack states (bosonic case $M = 0$ $J^{\alpha_{k,2}}_{\lambda^0(k,2)}$, fermionic case $M = 1$ $S^{\alpha_{k,2}}_{\lambda^0(k,2)+\delta}$). Such states are known to be densest ground states of $(k+1)$-body repulsion Hamiltonian. Among noted Jack states one can also find Gaffnian state. THe bosonic Gaffnian wave function may be expressed as $J^{\alpha_{2,3}}_{\lambda^0(2,3)}$ and in the fermionic case as $S^{\alpha_{2,3}}_{\lambda^0(2,3)+\delta}$. The Gaffnian is also characterized as a ground state of non physical four-body interaction Hamiltonian.

4. Results and conclusions: numerical generation of Jack polynomials

In the section 3 we discussed recursion formula for coefficients of Jack polynomials in monomial basis and fermionic Jack coefficients in Slater determinant basis. Both formulas can be adapted to the form of computer program

Partitions analyzed in the program may be stored as a vector of integers (we recommend storing partitions in occupational representation). There is a problem of generation of a list of all partitions of fixed weight, dominated by a selected partition λ, listed according to a reverse lexicographic order. It may be solved as follows: one generates a list of all partitions smaller than λ in the reverse lexicographic order and then remove partitions not dominated by the λ. When the list is obtained, one focuses on rather straightforward calculations of the eigenvalues and coefficients $C_{\lambda\mu}$ (bosonic case), $C^F_{\lambda\mu}$ (fermionic case). Next step is to execute recursion formula given in (84) and (89). Finally if one wishes to get coefficients of Jack wave function on a Haldane sphere one uses inverse of stereographic projection and map coefficients onto the sphere.

As an example of properly executed algorithm, we give results of numerical calculations of Jack polynomials in a form of Tables 1 and 2.

Table 1. Normalized coefficients of the Jack polynomial $J^{-2}_{(4,2,0)}(x,y,z)$ in the monomial basis. The coefficient are also expansion of three particle bosonic Laughlin wave function for $\nu = \frac{1}{2}$ state.

m_λ	coefficient
$m_{(4,2,0)}$	0.14285
$m_{(4,1,1)}$	−0.28571
$m_{(3,3,0)}$	−0.28571
$m_{(3,2,1)}$	0.28571
$m_{(2,2,2)}$	−0.85714

Table 2. Normalized coefficients of the Jack polynomial $J^{-\frac{2}{5}}_{(6,0)}(x,y)$ in the monomial basis. The coefficient are also expansion of three particle bosonic Laughlin wave function for $\nu = \frac{1}{6}$ state.

m_λ	coefficient
$m_{(6,0)}$	0.03886
$m_{(5,1)}$	−0.23319
$m_{(4,2)}$	0.58299
$m_{(3,3)}$	−0.77732

Both Tables 1 and 2 show coefficients of Jack wave function in small Hilbert spaces. However capabilities of computer program are greater and

one can generate Jacks in spaces of bigger dimensionality. In the Table 3 we examine overlaps of fermionic Jacks representing Laughlin $\nu = 1/3$ wave functions mapped onto the Haldane sphere and ground states of Coulomb interaction Hamiltonian, for different sizes of a system.

Table 3. Overlaps of the indicated of Jack states wave function $\nu = 1/3$ with LLL Coulomb ground states. Consecutive columns are electron number N, magnetic flux on the sphere $2Q$, dimension of the relevant N-body subspace, and the overlaps with Coulomb states in the LLL in GaAs.

N	$2Q$	dim	overlap
11	30	$1 \cdot 10^6$	0.9922
12	33	$8 \cdot 10^6$	0.9909
13	36	$4 \cdot 10^7$	0.9898
14	39	$3 \cdot 10^8$	0.9887

In conclusion we introduced basics of FQHE and standard tools used in its analysis like Haldane sphere or composite fermions. Then we followed with brief introduction to the theory of symmetric functions. We motivated adaptation of Jack polynomials in the context of FQHE and presented method of numerical obtaining of coefficients of Jacks and fermionic Jacks. As an illustration of discussed recursion formulas we gave both values of coefficients of Jack wave functions and overlaps of fermionic Jacks and Coulomb ground states. The paper should be treated as merely an introduction to the topic of Jack polynomials in FQHE. Not all advantages of Jack-based approach have been discussed. As we stated, coefficients of states related to Jack polynomials can be obtained with recursion formula rather than tedious Hamiltonian diagonalization. Moreover, the combinatorial structure of Jack polynomials is of use in analytical analysis. Definition of Jacks as eigenvectors of H^{CSM} can be a great tool, other properties like triangularity of expansion in monomial basis has not been explored yet. Moreover topic of more general symmetric functions in FQHE like Macdonald polynomials is not sufficiently examined.

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