Energy, interaction, and photoluminescence of spin-reversed quasielectrons in fractional quantum Hall systems

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The energy and photoluminescence spectra of a two-dimensional electron gas in the fractional quantum Hall regime are studied. The single-particle properties of reversed-spin quasielectrons (QE_R's) as well as the pseudopotentials of their interaction with one another and with Laughlin quasielectrons (QE's) and quasiholes (QH's) are calculated. Based on the short-range character of the QE_R–QE_R and QE_R–QE repulsion, the partially unpolarized incompressible states at the filling factors $\nu=\frac{4}{11}$ and $\frac{5}{13}$ are postulated within Haldane's hierarchy scheme. To describe photoluminescence, the family of bound $h(\mathrm{QE}_R)_n$ states of a valence hole h and n QE_R's are predicted in analogy to the found earlier fractionally charged excitons $h\mathrm{QE}_n$. The binding energy and optical selection rules for both families are compared. The $h\mathrm{QE}_R$ is found radiative in contrast to the dark hQE, and the $h(\mathrm{QE}_R)_2$ is found nonradiative in contrast to the bright $h\mathrm{QE}_2$.

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I. INTRODUCTION

The integer[1] and fractional[2–4] quantum Hall effects (IQHE and FQHE) both depend on the finite gap Δ for charge excitations that opens in a two-dimensional electron gas (2DEG) at the specific (integral or fractional) filling factors ν, defined as the number of electrons N divided by the Landau level (LL) degeneracy g. At sufficiently low temperatures, this gap makes the system incompressible and, among other effects, forbids electric conductance and causes quantization of Hall resistivity.

It is quite remarkable that the most prominent FQH states, so-called Laughlin ground states[3] that occur at $\nu=(2p+1)^{-1}$ (p is an integer), are the only ones that are maximally spin polarized solely due to the electron-electron exchange interaction. At other filling factors, the 2DEG is known[5–11] to be at least partially unpolarized unless the Zeeman energy E_Z is sufficiently large. Only partial polarization of the FQH states at the filling factors other than $\nu=(2p+1)^{-1}$ causes transitions[12] between incompressible and compressible or different incompressible phases as a function of E_Z, realized in tilted-field experiments.[13–16] The finite excitation gap Δ of the Laughlin state results from the finite energies ε of its elementary charge excitations, Laughlin quasielectrons (QE's) and quasiholes (QH's), as well as from the lack of the particle-hole symmetry between them that causes a magneto-roton type of dispersion of the QE-QH interaction with a minimum at a finite wave vector k. Indeed, the calculated[17–22] energy $\varepsilon_{\mathrm{QE}}+\varepsilon_{\mathrm{QH}}$ needed to create a spatially separated QE-QH pair necessary for electric current agreed reasonably well with the activation energy obtained from the temperature dependences of the FQHE at $\nu=(2p+1)^{-1}$.

Therefore, it was quite surprising when Rezayi[23] and Chakraborty *et al.*[24] discovered that another low-energy excitation of the Laughlin state exists, a spin-density wave, which becomes gapless at $E_Z=0$. It turns out that it is only due to a finite Zeeman energy that the spontaneous creation of spin waves, each consisting of a positively charged QH and a negatively charged reversed-spin quasielectron (QE_R), does not destroy incompressibility of Laughlin states in the experimental 2DEG systems. Although the spin excitations of Laughlin states have been extensively studied in the context of the real-space spin patterns called skyrmions[25,26] (particularly at $\nu=1$), our knowledge of their interaction with one another or with other excitations, or their optical properties is not yet complete (especially at fractional ν). In this paper we address both of these issues.

First, we identify QE, QH, and QE_R as the three elementary quasiparticles (QP's) of a Laughlin state and determine their mutual interaction pseudopotentials $V(\mathcal{R})$, defined[20] as the dependence of the pair interaction energy V on the relative pair angular momentum $\mathcal{R}$. For example, the QE_R-QE_R pseudopotential is found to be very different from the QE-QE pseudopotential at short range, which is the reason for incompressibility of a partially polarized $\nu=\frac{4}{11}$ state at low E_Z (in contrast to the compressible[27] fully polarized state at the same ν). A partially polarized $\nu=\frac{4}{11}$ state has been also recently proposed by Park and Jain[28] within a composite fermion[29–31] (CF) model. However, their interpretation of the $\nu=\frac{4}{11}$ as a mixed state of CF's with two and four attached vortices (fluxes) is not very accurate in a sense that the two additional vortices (fluxes) attached to each spin-reversed CF are not vortices of the many-body wave function expressed in terms of the same coordinates (fluxes of the same effective magnetic field) as the original two attached to each electron (to form CF's). The correct interpretation necessarily involves reapplication of the CF transformation to some of the original CF's (those in a partially filled reversed-spin LL), in analogy to the CF hierarchy proposed by Sitko *et al.*[32,33] and essentially equivalent[27] to Haldane's hierarchy.[17] Let us stress that it is the short range of the QE_R-QE_R repulsion shown here that justifies application of Haldane hierarchy to QE_R's (or, equivalently, spin-reversed CF's).

Second, in analogy to the fractionally charged excitons[34,35] (FCX's) consisting of a number of QE's of a spin-polarized 2DEG bound to a valence-band hole h, we discuss the possible formation and radiative recombination of similar complexes denoted as FCX_R's and containing one or more QE_R's bound to a hole. We find that different optical selection rules for FCX's and FCX_R's could allow optical detection of QE_R's in the 2DEG without need for direct polarization measurement.

II. MODEL

The properties of spin-reversed quasielectrons (QE_R) are studied by exact numerical diagonalization in an ideal 2DEG with zero width and no disorder. The magnetic field B is assumed to be sufficiently large (the cyclotron energy $\hbar\omega_c \propto B$ much larger than the interaction energy scale $e^2/\lambda \propto \sqrt{B}$, where λ is the magnetic length) that only the lowest LL need be considered. In order to describe an infinite planar system with 2D translational symmetry in a finite-size calculation we use Haldane's spherical geometry[17] in which the (finite) LL degeneracy $g=2S+1$ is controlled by the strength $2S$ of the magnetic monopole placed in the center of the sphere of radius R. The monopole strength $2S$ is defined in the units of flux quantum $\phi_0=hc/e$, so that $4\pi R^2 B=2S\phi_0$ and $R^2=S\lambda^2$. The single-particle states on a sphere $|S,l,m\rangle$ are called monopole harmonics.[17,21,36] They are eigenstates of length l and projection m of angular momentum and form LL's labeled by $n=l-S$, analogous to those of planar geometry. The lowest LL included in the present calculation has $n=0$ and $l=S$, and its orbitals are simply denoted by $|m\rangle$ with $|m|\leqslant S$. The electronic spin is included in the model by adding a quantum number σ denoting the projection of spin. As usual, the Zeeman term is taken as $E_Z \propto B\sigma$ to avoid an unphysical spin-orbit coupling resulting for $E_Z \propto \boldsymbol{B\sigma}$ and for a heterogeneous (radial) magnetic field on a sphere.

The many-electron interaction Hamiltonian reads

$$H_{ee}=\sum c^\dagger_{m_1\sigma}c^\dagger_{m_2\sigma'}c_{m_3\sigma'}c_{m_4\sigma}\langle m_1m_2|V_{ee}|m_3m_4\rangle, \quad (1)$$

where operators $c^\dagger_{m\sigma}$ and $c_{m\sigma}$ create and annihilate an electron in the state $|m\sigma\rangle$, the summations go over all orbital and spin indices, and the two-body interaction matrix elements are calculated for the Coulomb potential $V_{ee}(r)=e^2/r$. Hamiltonian H_{ee} is diagonalized in the basis of N-electron Slater determinants

$$|m_1\sigma_1\cdots m_N\sigma_N\rangle=c^\dagger_{m_1\sigma_1}\cdots c^\dagger_{m_N\sigma_N}|\mathrm{vac}\rangle, \quad (2)$$

where $|\mathrm{vac}\rangle$ stands for the vacuum state. While using basis (2) allows automatic resolution of two good many-body quantum numbers, projection of spin ($J_z=\Sigma\sigma_i$), and angular momentum ($L_z=\Sigma m_i$), the other two, length of spin (J) and angular momentum (L), are resolved numerically in the diagonalization of each appropriate (J_z,L_z) Hilbert subspace.

In order to describe the reversed-spin fractionally charged exciton (FCX_R) states, a single valence-band hole h is added to the model N-electron system. Since, as for FCX's, the formation of FCX_R states requires weakening of the electron-hole attraction compared to the electron-electron repulsion,[35] the hole is placed on a parallel plane, separated by a distance d (of the order of λ) from the 2DEG. Because the physics of an isolated FCX or FCX_R to a good approximation does not depend on the (possibly complicated) structure of the valence band, the single-hole wave functions are taken the same as for electrons (except for the reversed signs of m and σ). This means that both inter-LL hole scattering and the mixing between heavy- and light-hole subbands are ignored. The weak electron-hole exchange is also neglected so that the hole spin has no effect on the dynamics of an FCX or FCX_R, and the interaction of a hole with the 2DEG is described by the following spin-conserving term:

$$H_{eh}=\sum c^\dagger_{m_1\sigma}h^\dagger_{m_2}h_{m_3}c_{m_4\sigma}\langle m_1m_2|V_{eh}|m_3m_4\rangle \quad (3)$$

in the total Hamiltonian $H=H_{ee}+H_{eh}$. In the above, operators $h^\dagger_m$ and h_m create and annihilate a hole in the orbital $|m\rangle$ of the valence band, and the electron-hole interaction is defined by the Coulomb potential $V_{eh}(r)=-e^2/\sqrt{r^2+d^2}$. The exclusion of the hole-hole interaction effects from H reflects the fact that $\nu_h \ll \nu$. Interaction Hamiltonian H is diagonalized in the basis of single-particle configurations

$$|m_1\sigma_1\cdots m_N\sigma_N;m_h\rangle=c^\dagger_{m_1\sigma_1}\cdots c^\dagger_{m_N\sigma_N}h^\dagger_{m_h}|\mathrm{vac}\rangle, \quad (4)$$

and the set of good quantum numbers labeling many-electron–one-hole eigenstates includes J_z and J of the electrons, hole spin σ_h (omitted in our equations), and the length (L), and projection (L_z) of angular momentum of the total electron-hole system.

The justification for using Haldane's spherical geometry to model an infinite planar 2DEG (with or without additional valence holes) relies on the exact mapping between the orbital numbers L and L_z and the two good quantum numbers on a plane (resulting from the 2D translational symmetry), angular momentum projection $\mathcal{M}$ and an additional angular momentum quantum number $\mathcal{K}$ associated with partial decoupling of the center-of-mass motion of an electron-hole system in a homogeneous magnetic field.[37,38] This mapping guarantees correct description of such symmetry-dependent effects as degeneracies in the energy spectrum or the optical selection rules (associated with conservation of $\mathcal{M}$ and $\mathcal{K}$ or L and L_z in the absorption or emission of a photon). The energy values obtained on a sphere generally depend on the surface curvature, that is on $R/\lambda=\sqrt{S}$. However, for those

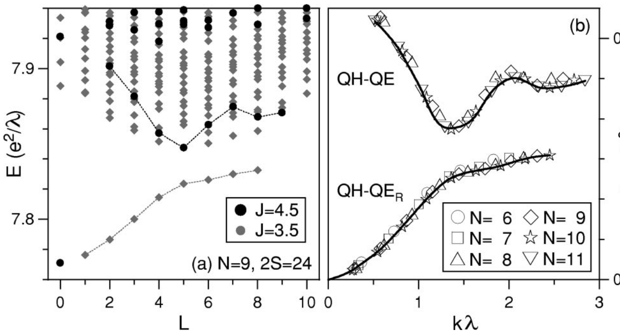

FIG. 1. (a) The energy spectrum (Coulomb energy E versus angular momentum L) of the system of $N=9$ electrons on Haldane sphere at the monopole strength $2S=3(N-1)=24$. Black dots and gray diamonds mark states with the total spin $J=\frac{1}{2}N=\frac{9}{2}$ (maximum polarization) and $J=\frac{1}{2}N-1=\frac{7}{2}$ (one reversed spin), respectively. Ground state is the Laughlin $\nu=\frac{1}{3}$ state. Lines connect states containing one QE-QH ($J=\frac{9}{2}$) or QE$_R$-QH ($J=\frac{7}{2}$) pair. (b) The dispersion curves (excitation energy $\mathcal{E}_\Sigma=E-E_0$ versus wave vector k) for the $\Sigma=0$ charge-density wave (QE-QH pair) and the $\Sigma=1$ spin-density wave (QE$_R$-QH pair) in the Laughlin $\nu=\frac{1}{3}$ ground state, calculated in the systems of $N\leqslant 11$ electrons on Haldane sphere. λ is the magnetic length.

energies that describe finite-size objects (such as QE$_R$ or FCX$_R$ studied here) or their interaction at a finite range (here, pseudopotential parameters for interaction of QE$_R$ with other particles), the values characteristic of an infinite planar system can be estimated from the calculation done for sufficiently large $2S$ and N (or extrapolation of finite-size data to the $2S\to\infty$ limit).

III. SPIN-REVERSED QUASIELECTRONS: RESULTS AND DISCUSSION

A. Stability and single-particle properties

It is well known that even in the absence of the Zeeman energy gap $E_Z=0$, the ground state of the 2DEG in the lowest LL is completely spin-polarized at the precise values of the Laughlin filling factor $\nu=(2p+1)^{-1}$, with $p=0,1,2,\ldots$. There are two types of elementary charge-neutral excitations of Laughlin $\nu=(2p+1)^{-1}$ ground states, carrying spin $\Sigma=0$ or 1, respectively. Their dispersion curves (energy as a function of wave vector) $\mathcal{E}_\Sigma(k)$, have been studied for all combinations of p and Σ. While the formulas for the $\nu=1$ ground state have been evaluated analytically,[39–41] in Fig. 1 we present the exact numerical results for $\nu=\frac{1}{3}$ obtained from our exact diagonalization of up to $N=11$ electrons on Haldane's sphere. As an example, in Fig. 1(a), we show the entire low-energy spectrum of an $N=9$ system with all spins polarized and with one reversed spin (Hilbert subspaces of total spin $J=\frac{1}{2}N-\Sigma=\frac{9}{2}$ and $\frac{7}{2}$ for $\Sigma=0$ and 1, respectively), from which the dispersion curves $\mathcal{E}_\Sigma(k)$ are obtained. The energy E is plotted as a function of angular momentum L, and $2S=3(N-1)=24$ is the strength of the magnetic monopole inside Haldane's sphere corresponding to the LL degeneracy $g=2S+1=25$ and the Laughlin filling factor $\nu=(N-1)/(g-1)=\frac{1}{3}$ (for the details of Haldane's spherical geometry see Refs. 17,21,36). The energy E does not include the Zeeman term E_Z, which scales differently than the plotted Coulomb energy with the magnetic field B. The excitation energies $\mathcal{E}_\Sigma=E-E_0$ (where E_0 is the Laughlin ground state energy) have been calculated for the states identified in the finite-size spectra as the $\Sigma=0$ charge-density wave and the $\Sigma=1$ spin-density wave. These states are marked with dotted lines in Fig. 1(a). The values of $\mathcal{E}_\Sigma$ obtained for different $N\leqslant 11$ have been plotted together in Fig. 1(b) as a function of the wave vector $k=L/R=(L/\sqrt{S})\lambda^{-1}$. Clearly, using the appropriate units of λ^{-1} for wave vector and e^2/λ for excitation energy in Fig. 1(b) results in the quick convergence of the curves with increasing N, and allows an accurate prediction of the dispersion curves in an infinite system, as marked with thick lines. The most significant features of these curves are (i) the finite gap $\Delta_0\approx 0.076\,e^2/\lambda$ and the magnetoroton minimum $k\approx 1.5\lambda^{-1}$ in $\mathcal{E}_0(k)$ and (ii) the vanishing of $\mathcal{E}_1(k)$ in the $k\to 0$ limit (for $E_Z=0$).

The similar nature of the charge and spin waves in the $\nu=\frac{1}{3}$ state to those at $\nu=1$ lies at the heart of the composite fermion (CF) picture,[29–31] in which these excitations correspond to promoting one CF from a completely filled lowest ($n=0$) spin-$\downarrow$ CF LL either to the first excited ($n=1$) CF LL of the same spin ($\downarrow$) or to the same CF LL ($n=0$) but with the reversed spin ($\uparrow$). The three constituent QP's of which the charge and spin waves are composed: a hole in the $n=0$ spin-$\downarrow$ CF LL and particles in the $n=1$ spin-$\downarrow$ and $n=0$ spin-$\uparrow$ CF LL's, are analogous to those in the electron LL's from which the charge and spin waves at $\nu=1$ are built.

Independently of the CF picture, one can define three types of QP's (elementary excitations) of the Laughlin $\nu=\frac{1}{3}$ fluid. They are Laughlin quasiholes (QH's) and quasielectrons (QE's) and Rezayi spin-reversed quasielectrons (QE$_R$). The excitations in Fig. 1 are more complex in a sense that they consist of a (neutral) pair of QH and either QE ($\Sigma=0$) or QE$_R$ ($\Sigma=1$). Each of the QP's is characterized by such single-particle quantities as (fractional) electric charge ($\mathcal{Q}_{\rm QH}=+\frac{1}{3}e$ and $\mathcal{Q}_{\rm QE}=\mathcal{Q}_{{\rm QE}_R}=-\frac{1}{3}e$), energy $\varepsilon_{\rm QP}$, or degeneracy $g_{\rm QP}$ of the single-particle Hilbert space. On Haldane's sphere, the degeneracy $g_{\rm QP}$ is related to the angular momentum $l_{\rm QP}$ by $g_{\rm QP}=2l_{\rm QP}+1$, with $l_{\rm QH}=l_{{\rm QE}_R}=S^*$ and $l_{\rm QE}=S^*+1$ and $2S^*=2S-2(N-1)$ being the effective monopole strength in the CF model.

The energies $\varepsilon_{\rm QP}$ to create an isolated QP of each type in the Laughlin ground state have been previously estimated in a number of ways. Here, we present our results of exact diagonalization calculation for $N\leqslant 11$ ($\varepsilon_{\rm QE}$ and $\varepsilon_{\rm QH}$) and $N\leqslant 10$ ($\varepsilon_{{\rm QE}_R}$). In Fig. 2(a) we show an example of the numerical energy spectrum for the system of $N=9$ electrons, in which an isolated QE or QE$_R$ occurs at $2S=3(N-1)-1=23$ in the subspace of $J=\frac{1}{2}N=\frac{9}{2}$ and $J=\frac{1}{2}N-1=\frac{7}{2}$, respectively. Both of these states have been identified in Fig. 2(a). To estimate $\varepsilon_{\rm QE}$ and $\varepsilon_{{\rm QE}_R}$, we use the standard procedure[19–22,27] to take into account the finite-size effects (the dependence of λ on $2S$, $S\lambda^2=R^2$), and express the energies E of Fig. 2(a) in the units of e^2/λ with λ appropriate for $\nu=\frac{1}{3}$, before subtracting from them the Laughlin

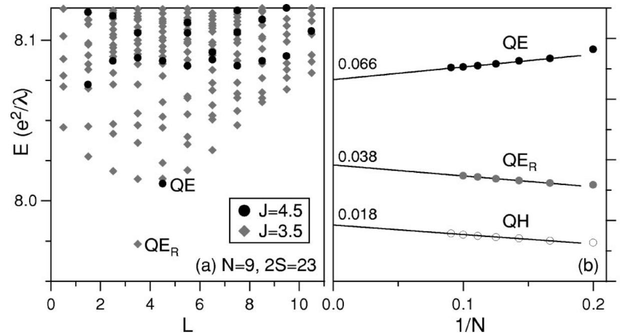

FIG. 2. (a) The energy spectrum (Coulomb energy E versus angular momentum L) of the system of $N=9$ electrons on Haldane sphere at the monopole strength $2S=3(N-1)-1=23$. Black dots and gray diamonds mark states with the total spin $J=\frac{1}{2}N=\frac{9}{2}$ (maximum polarization) and $J=\frac{1}{2}N-1=\frac{7}{2}$ (one reversed spin), respectively. Ground state at $J=\frac{7}{2}$ is the reversed-spin quasielectron QE$_R$ of the Laughlin $\nu=\frac{1}{3}$ fluid, and the lowest-energy state at $J=\frac{9}{2}$ is the Laughlin quasielectron QE. (b) The energies ε of all three types of quasiparticles of the Laughlin $\nu=\frac{1}{3}$ ground state (QH, QE, and QE$_R$) calculated in the systems of $N\leqslant 11$ electrons on Haldane sphere and plotted as a function of N^{-1}. The numbers give the results of linear extrapolation to an infinite (planar) system. λ is the magnetic length.

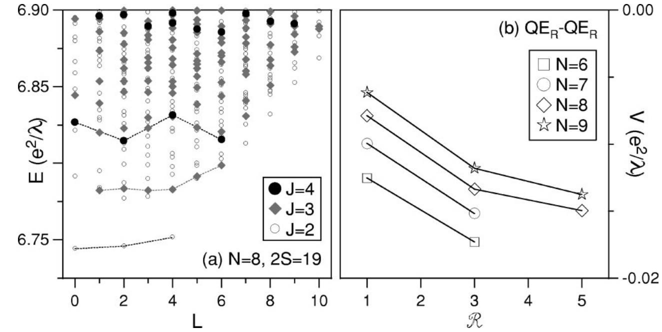

FIG. 3. (a) The energy spectrum (Coulomb energy E versus angular momentum L) of the system of $N=8$ electrons on Haldane sphere at the monopole strength $2S=3(N-1)-2=19$. Black dots, gray diamonds, and open circles mark states with the total spin $J=\frac{1}{2}N=4$ (maximum polarization), $J=\frac{1}{2}N-1=3$ (one reversed spin), and $J=\frac{1}{2}N-2=2$ (two reversed spins), respectively. Lines connect states containing one QE-QE ($J=4$), QE-QE$_R$ ($J=3$), or QE$_R$-QE$_R$ ($J=2$) pair. (b) The pseudopotentials (pair energy V versus relative angular momentum $\mathcal{R}$) of the QE$_R$-QE$_R$ interaction calculated in the systems of $N\leqslant 9$ electrons on Haldane sphere. λ is the magnetic length.

ground state energy of Fig. 1(a). Plotting the results for different values of N in Fig. 2(b) as a function of N^{-1} allows the extrapolation to an infinite system, with the limiting values of $\varepsilon_{\rm QE}=0.0664\,e^2/\lambda$ and $\varepsilon_{{\rm QE}_R}=0.0383\,e^2/\lambda$ (with the difference $\varepsilon_{\rm QE}-\varepsilon_{{\rm QE}_R}=0.0281\,e^2/\lambda$ in remarkable agreement with Rezayi's original estimate[23] based on his numerics for $N\leqslant 6$). For completeness, we have also plotted the QH energies, which extrapolate to $\varepsilon_{\rm QH}=0.0185e^2/\lambda$. Note that to obtain the so-called "proper" QP energies in a finite system,[19,21,22] $\tilde{\varepsilon}_{\rm QP}(N)$, the term $\mathcal{Q}_{\rm QP}^2/2R$ must be added to each value in Fig. 2(b). The linear extrapolation of $\tilde{\varepsilon}_{\rm QP}(N)$ to $N^{-1}\to 0$ gives $\tilde{\varepsilon}_{\rm QE}=0.0737\,e^2/\lambda$, $\tilde{\varepsilon}_{{\rm QE}_R}=0.0457\,e^2/\lambda$, and $\tilde{\varepsilon}_{\rm QH}=0.0258\,e^2/\lambda$. The energies of spatially separated QE-QH and QE$_R$-QH pairs (activation energies in transport experiments) are hence equal to $\mathcal{E}_0(\infty)=\tilde{\varepsilon}_{\rm QE}+\tilde{\varepsilon}_{\rm QH}=0.0995\,e^2/\lambda$ and $\mathcal{E}_1(\infty)=\tilde{\varepsilon}_{{\rm QE}_R}+\tilde{\varepsilon}_{\rm QH}=0.0715\,e^2/\lambda$.

While the QH's are the only type of QP's that occur in low-energy states at $\nu<(2p+1)^{-1}$, the QE's and QE$_R$'s are two competing excitations at $\nu>(2p+1)^{-1}$. As pointed out by Rezayi[23] and Chakraborty *et al.*,[24] whether QE's or QE$_R$'s will occur at low energy depends on the relation between their energies including the Zeeman term, $\varepsilon_{\rm QE}$ and $\varepsilon_{{\rm QE}_R}+E_Z$. Although it is difficult to accurately estimate the value of E_Z in an experimental sample because of its dependence on a number of factors (material parameters, well width w, density ϱ, magnetic field B, etc.), it seems that both scenarios with QE's and QE$_R$'s being lowest-energy QP's are possible. For example, using the bulk value for the effective g^* factor in GaAs ($dE_Z/dB=0.03$ meV/T) results in the QE$_R$-QE crossing at $B=18$ T, while including the dependence of g^* on w and B as described in Ref. 42 makes QE$_R$ more stable than QE up to $B\sim 100$ T.

B. Interaction with other quasiparticles

Once it is established which of the QP's occur at low energy in a particular system (defined by ϱ, w, B, ν, etc.), their correlations can be understood by studying the appropriate pair interaction pseudopotentials.[20,22,33,43] The pseudopotential $V(\mathcal{R})$ is defined[20] as the dependence of pair interaction energy V on relative orbital angular momentum $\mathcal{R}$. On a plane, $\mathcal{R}$ for a pair of particles ab is the angular momentum associated with the (complex) relative coordinate $z=z_a-z_b$. On Haldane's sphere, the compatible definition of $\mathcal{R}$ depends on the sign of $\mathcal{Q}_a\mathcal{Q}_b$: for a pair of opposite charges, $\mathcal{R}$ is the length of total pair angular momentum, $L=|\mathbf{l}_a+\mathbf{l}_b|$, while for two charges of the same sign, $\mathcal{R}=l_a+l_b-L$. In all cases, $\mathcal{R}\geqslant 0$ and larger $\mathcal{R}$ corresponds to a larger average ab separation.[22,33] Furthermore, only odd values of $\mathcal{R}$ are allowed for indistinguishable ($a=b$) fermions.

Since the QE-QH and QE$_R$-QH pseudopotentials have been plotted in Fig. 1 ($V_{\rm QE\text{-}QH}=\mathcal{E}_0$ and $V_{{\rm QE}_R\text{-QH}}=\mathcal{E}_1$), and the QE-QE and QH-QH pseudopotentials can be found for example in Ref. 27, we only need to discuss $V_{{\rm QE}_R\text{-}{\rm QE}_R}$ and $V_{{\rm QE}\text{-}{\rm QE}_R}$. Two QE$_R$'s occur in an N-electron system with at least two reversed spins ($J\leqslant\frac{1}{2}N-2$) and at $2S=3(N-1)-2$ (i.e., at $g=g_0-2$ where g_0 corresponds to the Laughlin state). An example of the energy spectrum is shown in Fig. 3(a) for $N=8$ at $2S=19$. The lowest-energy states in the subspaces of $J=\frac{1}{2}N=4$, $\frac{1}{2}N-1=3$, and $\frac{1}{2}N-2=2$ are connected with dashed lines and contain a QE-QE, QE-QE$_R$, and QE$_R$-QE$_R$ pair, respectively. The angular momenta L that occur in these bands result from addition of $\mathbf{l}_{\rm QE}$ and/or $\mathbf{l}_{{\rm QE}_R}$ (with $l_{\rm QE}=S^*+1=\frac{7}{2}$ and $l_{{\rm QE}_R}=S^*=\frac{5}{2}$). For identical fermions, the addition must be followed by antisymmetrization that picks out only odd values of $\mathcal{R}$ for the QE-QE and QE_R-QE_R pairs.

An immediate conclusion from Fig. 3(a) is that the maximally spin-polarized ($J=\frac{1}{2}N$) system is unstable at the filling factor close but not equal to the Laughlin value of $\nu=\frac{1}{3}$ (the actual spin polarization decreases with decreasing E_Z, and $J=0$ for $E_Z=0$). This was first pointed out by Rezayi[23] and interpreted in terms of an effective attraction between $\Sigma=1$ spin waves; in this paper we prefer to use charged QP's as the most elementary excitations and explain the observed ordering of different J bands by the fact that $\varepsilon_{\mathrm{QE}}\neq\varepsilon_{\mathrm{QE}_R}$ (at $E_Z=0$, $\varepsilon_{\mathrm{QE}}$-$\varepsilon_{\mathrm{QE}_R}\approx 0.0281\,e^2/\lambda$) and the particular form of involved interaction pseudopotentials (see further in the text).

We have calculated the QE-QE_R and QE_R-QE_R pseudopotentials from the energy spectra as that in Fig. 3(a) by converting L into $\mathcal{R}$ and subtracting the Laughlin ground state energy and the energy of two appropriate QP's from the total N-electron energy, $V_{AB}(\mathcal{R})=E(L)-E_0-\varepsilon_A-\varepsilon_B$. To minimize the finite-size effects, all subtracted energies are given in the same units of e^2/λ_0, where $\lambda_0=R/\sqrt{S_0}$ corresponds to $2S_0=3(N-1)$, i.e., to $\nu=\frac{1}{3}$. The result for $V_{\mathrm{QE}_R\text{-}\mathrm{QE}_R}$ and $N\leqslant 9$ is shown in Fig. 3(b). Clearly, obtained values of $V_{\mathrm{QE}_R\text{-}\mathrm{QE}_R}(\mathcal{R})$ still depend on N and, for example, the positive sign characteristic of repulsion between equally charged particles is only restored in the $N^{-1}\to 0$ limit with $V_{\mathrm{QE}_R\text{-}\mathrm{QE}_R}(1)$ of the order of 0.01 e^2/λ (compare with discussion of the signs of $V_{\text{QE-QE}}$ and $V_{\text{QH-QH}}$ in Ref. 44). However, it seems that the monotonic character of $V_{\mathrm{QE}_R\text{-}\mathrm{QE}_R}(\mathcal{R})$ is independent of N. More importantly, $V_{\mathrm{QE}_R\text{-}\mathrm{QE}_R}$ is also a superlinear function of $L(L+1)$. This implies[22,33,43] Laughlin correlations and incompressibility at $\nu_{\mathrm{QE}_R}=(2p+1)^{-1}$, in analogy to the spin-polarized Laughlin states of QE's or QH's in Haldane's hierarchy picture.[17,27] The most prominent of QE_R Laughlin states, $\nu_{\mathrm{QE}_R}=\frac{1}{3}$, corresponds to the electronic filling factor of $\nu=\frac{4}{11}$ and the 75% spin polarization ($J=\frac{1}{4}N$). This state has been first suggested by Beran and Morf.[45] Since the $\nu_{\mathrm{QE}}=\frac{1}{3}$ state is compressible,[27] the experimental observation[46] of the FQHE at $\nu=\frac{4}{11}$ seems to prove the formation of QE_R's in the $\nu=\frac{1}{3}$ state without need for direct measurement of spin polarization. The expected critical dependence of the excitation gap at $\nu=\frac{4}{11}$ on the Zeeman gap E_Z might be revealed in tilted-field experiments. This dependence will be very different than at some other fractions. For example, the fact that incompressibility at $\nu=\frac{2}{5}$ can be a result of either maximally spin-polarized $\nu_{\mathrm{QE}}=1$ or completely spin-unpolarized ($J=0$) $\nu_{\mathrm{QE}_R}=1$ state gives rise to FQHE at this filling in both small and large E_Z regime. On the other hand, spin-unpolarized FQHE is not expected in the $\frac{1}{4}<\nu<\frac{1}{3}$ range (because spin-reversed QH's in the $\nu=\frac{1}{3}$ state do not exist), and the $\nu=\frac{2}{7}$ and $\frac{4}{13}$ states (corresponding[27] to $\nu_{\mathrm{QH}}=\frac{1}{3}$ and $\frac{1}{5}$) should remain incompressible and compressible, respectively, over a wide range of E_Z.

The QE-QE_R pseudopotentials were calculated from similar spectra as that of $J=3$ in Fig. 3(a). As another example, in Fig. 4(a) we show the spectrum for $N=9$, in which only two values of $J=\frac{1}{2}N=\frac{9}{2}$ and $\frac{1}{2}N-1=\frac{7}{2}$ have been included. The lowest energy states in these two J subspaces (connected with dashed lines) contain a QE-QE and QE-QE_R pair, respectively. Using the same procedure as for $V_{\mathrm{QE}_R\text{-}\mathrm{QE}_R}$, we have calculated $V_{\text{QE-}\mathrm{QE}_R}(\mathcal{R})$. The results for $N\leqslant 10$ are presented in Fig. 4(b). As for $V_{\mathrm{QE}_R\text{-}\mathrm{QE}_R}$ in Fig. 3(b), the values of $V_{\text{QE-}\mathrm{QE}_R}(\mathcal{R})$ calculated in a finite system depend on N. The values extrapolated to the $N^{-1}\to 0$ limit are also similar, with $V_{\text{QE-}\mathrm{QE}_R}(0)\to 0.015\,e^2/\lambda$ and $V_{\text{QE-QE}R}(1)\to 0.01\,e^2/\lambda$.

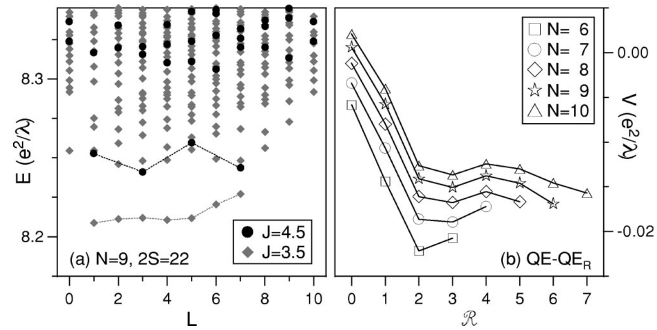

FIG. 4. (a) The energy spectrum (Coulomb energy E versus angular momentum L) of the system of $N=9$ electrons on Haldane sphere at the monopole strength $2S=3(N-1)-2=22$. Black dots and gray diamonds mark states with the total spin $J=\frac{1}{2}N=\frac{9}{2}$ (maximum polarization) and $J=\frac{1}{2}N-1=\frac{7}{2}$ (one reversed spin), respectively. Lines connect states containing one QE-QE ($J=\frac{9}{2}$) or QE-QE_R ($J=\frac{7}{2}$) pair. (b) The pseudopotentials (pair energy V versus relative angular momentum $\mathcal{R}$) of the QE-QE_R interaction calculated in the systems of $N\leqslant 10$ electrons on Haldane sphere. λ is the magnetic length.

Despite finite-size errors, the comparison of the curves for $N\leqslant 10$ is sufficient to notice quite different behavior of $V_{\text{QE-}\mathrm{QE}_R}(\mathcal{R})$ from both $V_{\mathrm{QE}_R\text{-}\mathrm{QE}_R}(\mathcal{R})$ and $V_{\text{QE-QE}}(\mathcal{R})$. Two important features of the $V_{\text{QE-}\mathrm{QE}_R}$ pseudopotential can be established: (i) the QE-QE_R repulsion is relatively strong at $\mathcal{R}\leqslant 1$ (short range) and saturates at larger $\mathcal{R}$, and (ii) $V_{\text{QE-}\mathrm{QE}_R}$ is superlinear in $L(L+1)$ only at $1\leqslant\mathcal{R}\leqslant 3$, but sublinear at $0\leqslant\mathcal{R}\leqslant 2$ and at larger $\mathcal{R}$. As a consequence, the short-range criterion[22,33,43] applied to $V_{\text{QE-}\mathrm{QE}_R}$ yields Laughlin correlations for QE-QE_R pairs only at $m=2$. The term ''Laughlin correlations'' used here is generally defined[20,22,43] as a tendency to avoid pair states with $\mathcal{R}$ smaller than certain m. At $\nu\leqslant m^{-1}$, these correlations are described by a Jastrow prefactor $\Pi_{ij}(x_i-y_j)^m$ in the many-body wave function (x and y are complex QE and QE_R coordinates, respectively).

Although it is not clear if QE's and QE_R's could coexist in the $\nu=\frac{1}{3}$ ''parent'' state in an experimental system (such mixed state would be sensitive to the value of E_Z), one can ask if such two-component QE-QE_R plasma could also be incompressible. This question can be answered within the generalized CF model[33,47] for all allowed combinations of Jastrow exponents $[m_{\text{QE-QE}}, m_{\mathrm{QE}_R\text{-}\mathrm{QE}_R}, m_{\text{QE-}\mathrm{QE}_R}]$. In this model, the reduced (effective) LL degeneracies of QP's are given by $g^*_{\mathrm{QE}}=g_{\mathrm{QE}}-(m_{\text{QE-QE}}-1)(N_{\mathrm{QE}}-1)-m_{\text{QE-}\mathrm{QE}_R}N_{\mathrm{QE}_R}$

and $g^*_{\mathrm{QE}_R}=g_{\mathrm{QE}_R}-(m_{\mathrm{QE}_R\text{-}\mathrm{QE}_R}-1)(N_{\mathrm{QE}_R}-1)-m_{\mathrm{QE}\text{-}\mathrm{QE}_R}N_{\mathrm{QE}}$, and the incompressibility condition is $N_{\mathrm{QP}}=g^*_{\mathrm{QP}}$ for both QE's and QE_R's. In the above, g_{QP} is the LL degeneracy of electrons and N_{QP} denotes the number of QP's of each type. It turns out that because the three involved QP pseudopotentials are not generally superlinear in $L(L+1)$, only few combinations of exponents $[m_{\mathrm{QE}\text{-}\mathrm{QE}},m_{\mathrm{QE}_R\text{-}\mathrm{QE}_R},m_{\mathrm{QE}\text{-}\mathrm{QE}_R}]$ are allowed, and of those only [1,1,2] satisfies the incompressibility condition. The hypothetical [1,1,2] state of the QE-QE_R fluid corresponds to $\nu=\frac{5}{13}$ and 80% polarization ($J=\frac{3}{10}N$). Finite realizations of this state on Haldane's sphere occur for $N=5q+4$ ($q\geqslant 1$) at $2S=13q+7$, and have $N_{\mathrm{QE}}=q$ and $N_{\mathrm{QE}_R}=q+2$, which yields $J=\frac{3}{2}q$.

C. Optical properties

Once the single-particle energies ε and the two-particle interaction pseudopotentials $V(\mathcal{R})$ of all three types of QP's have been calculated, let us now turn to their optical properties. The effect of QE's on the photoluminescence (PL) spectrum of the Laughlin fluid has been studied in great detail.[33,35] The crucial facts are (i) the PL spectrum can be understood in terms of QE's and their interaction with one another and with a valence-band hole (h) only in the "weak-coupling regime" in which the electron-electron repulsion is sufficiently weak compared to the electron-hole attraction; this is realized in "asymmetric" structures in which the electron and hole layers are separated by a finite distance d (of the order of λ). (ii) In this regime, a positively charged h can bind one or two QE's to form "fractionally charged excitons" (FCX), hQE or $h\mathrm{QE}_2$. (iii) The 2D translational invariance results in orbital selection rules for the radiative recombination of FCX's; it turns out that the only bright states are $h\mathrm{QE}^*$ (an excited state of the dark hQE) and $h\mathrm{QE}_2$.

In analogy, we expect that a valence hole h could also form bound states with one or more QE_R's, denoted by FCX_R. However, unlike for FCX's, the stability of FCX_R complexes should depend on the Zeeman energy, the binding of more than one QE_R should be more difficult due to the stronger QE_R-QE_R repulsion, different angular momenta of QE and QE_R should result in different optical selection rules of FCX_R, and the possible annihilation of a hole with a reversed-spin electron should cause different polarization of FCX_R emission. To study the possible binding of FCX_R's we begin with the h-QE_R pseudopotential, shown in Fig. 5(a) for a $7e$-h system in which a hole interacts with $N=7$ electrons and for a few different values of d/λ. The values of $2S=3(N-1)-1=17$ and $J=\frac{1}{2}N-1=\frac{5}{2}$ are chosen so that one QE_R is present in the Laughlin $\nu=\frac{1}{3}$ state and interacts with the hole. In the CF picture of this configuration $2S^*=2S-2(N-1)=5$ so that six CF's fill completely the lowest CF LL of $g^*=2S^*+1$, leaving the seventh CF in the reversed-spin LL. The filled LL is incompressible, and only the single reversed-spin CF (i.e., QE_R) correlates with the hole. The $V_{h\text{-}\mathrm{QE}_R}$ is plotted as a function of the pair angular momentum whose values ($6\leqslant L\leqslant 11$) result from addition of $l_h=S$ and $l_{\mathrm{QE}_R}=S^*$. To ensure that exactly one QE_R is present in the Laughlin fluid and interacts with the hole at an

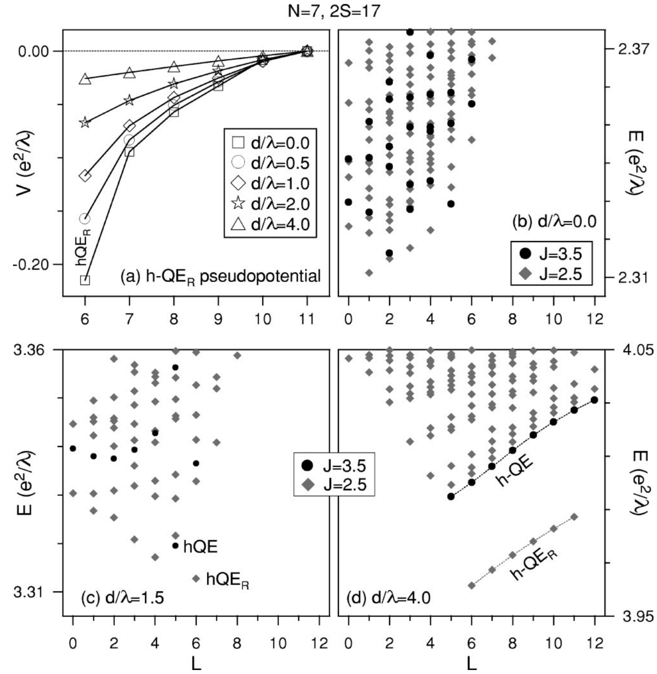

FIG. 5. (a) The pseudopotentials (pair energy V versus pair angular momentum L) of the h-QE_R interaction calculated in the system of $N=7$ electrons and one valence hole (h) on Haldane sphere at the monopole strength $2S=3(N-1)-1=17$. Different symbols correspond to different separations d between the electron and hole planes. (bcd) The energy spectra (Coulomb energy E versus angular momentum L) of the same, seven-electron–one-hole system at $2S=17$ at three different values of d. Black dots and gray diamonds mark states with the total electron spin $J=\frac{1}{2}N=\frac{7}{2}$ (maximum polarization) and $J=\frac{1}{2}N-1=\frac{5}{2}$ (one reversed spin), respectively. Lines in (d) connect states containing one h-QE ($J=\frac{7}{2}$) or h-QE_R ($J=\frac{5}{2}$) pair. The lowest-energy $J=\frac{7}{2}$ and $\frac{5}{2}$ states in (c) are the fractionally charged excitons, hQE and $h\mathrm{QE}_R$, respectively. λ is the magnetic length.

arbitrary (small) value of d, a special procedure[35] has been used in which the electric charge of the hole is reduced to $e/\epsilon\ll e$. Clearly, the decrease of $V_{h\text{-}\mathrm{QE}_R}$ with a decrease of L (average h-QE_R separation) indicates h-QE_R attraction. The strength of this attraction, that is the binding energy $\Delta_{h\mathrm{QE}_R}\sim|V_{h\text{-}\mathrm{QE}_R}(l_h-l_{\mathrm{QE}_R})|$, depends on d and is similar to $\Delta_{h\mathrm{QE}}$; compare with Ref. 35. Therefore, in analogy to the QE case, we expect that bound $h\mathrm{QE}_R$ states will occur in a system containing free QE_R's at the values of d at which $\Delta_{h\mathrm{QE}}$ and $\Delta_{h\mathrm{QE}_R}$ is smaller than the Laughlin gap to create additional QE-QH pairs (note that since the projection J_z of the total electron spin is conserved at any d, neither FCX nor FCX_R couples to virtual QE_R-QH excitations).

In order to verify the above hypothesis, we have calculated the $7e$-h energy spectra with up to one reversed spin ($J=\frac{1}{2}N=\frac{7}{2}$ and $J=\frac{1}{2}N-1=\frac{5}{2}$). The results for $d/\lambda=0$, 1.5, and 4 are presented in Fig. 5(bcd). As expected, the $h\mathrm{QE}_R$ ground state develops together with the spin-polarized hQE state at d larger than about λ. The energy difference between $h\mathrm{QE}_R$ and hQE states at $d/\lambda=1.5$ is only about 0.007 e^2/λ, which is small compared to $\varepsilon_{\mathrm{QE}}-\varepsilon_{\mathrm{QE}_R}$. This is because hQE couples stronger than $h\mathrm{QE}_R$ to virtual QE-QH pair excitations of the underlying Laughlin state (QE-QE_R repulsion at short range is stronger than QE-QE repulsion). At d much larger than λ, the lowest energy states in Fig. 5(d) contain well defined h-QE or h-QE_R pairs with all possible values of L. The coupling to the virtual QE-QH excitations is reduced, and the h-QE_R and h-QE bands are separated by about the single-particle gap $\varepsilon_{\mathrm{QE}}$-$\varepsilon_{\mathrm{QE}_R}$.

To compare the optical properties of $h\mathrm{QE}$ and $h\mathrm{QE}_R$, it is essential to notice that, because $l_{\mathrm{QE}_R}\neq l_{\mathrm{QE}}$, also $l_{h\mathrm{QE}_R}=l_h-l_{\mathrm{QE}_R}=N-1$ is different from $l_{h\mathrm{QE}}=l_h-l_{\mathrm{QE}}=N-2$. The orbital selection rule for radiative recombination of bound FCX or FCX_R states results from the fact that an annihilated, optically active electron-hole pair carries no angular momentum.[35,42,37,38] Therefore, the angular momenta of the initial (bound) state and a final state in the emission process must be equal. On the other hand, it is known[34,35] that only those emission processes with minimum number of QP's involved can have significant spatial overlap with an initial (bound) state of small size, and thus significant intensity (oscillator strength τ^{-1}). Thus, $h\mathrm{QE}$ or $h\mathrm{QE}_R$ must both recombine to leave two QH's in the final state (and no additional QE-QH or QE_R-QH pairs). The allowed angular momenta of two identical QH's [in the final, $(N-1)e$ system] each with $l_{\mathrm{QH}}=\frac{1}{2}N$ are $L_{2\mathrm{QH}}=N-\mathcal{R}_{\mathrm{QH}}$, where $\mathcal{R}_{\mathrm{QH}}$ is an odd integer. The comparison of $L_{2\mathrm{QH}}$ with $l_{h\mathrm{QE}}$ and $l_{h\mathrm{QE}_R}$ makes it clear that, in contrast to the dark $h\mathrm{QE}$, the $h\mathrm{QE}_R$ ground state is radiative. Since $h\mathrm{QE}_R$ is the simplest of all FCX_R's and bright at the same time, its emission is expected to dominate the PL spectrum of a Laughlin fluid at $\nu>\frac{1}{3}$, in which free QE_R's are present. The larger FCX_R complexes, $h(\mathrm{QE}_R)_2$ and $h\mathrm{QE}_R\mathrm{QE}$ are also found in the numerical calculation at $d>\lambda$ (see Fig. 6), but being less strongly bound (due to larger QE_R-QE_R and QE_R–QE repulsion at short range) they are not expected to form as easily as $h\mathrm{QE}_2$ does in a spin-polarized system. Moreover, $h(\mathrm{QE}_R)_2$ turns out dark, and the formation of $h\mathrm{QE}_R\mathrm{QE}$ requires the presence of both QE's and QE_R's in the unperturbed electron system, which further limits the contribution of these bound states to the PL spectrum. Let also add that since $h\mathrm{QE}_R$ emits by recombination of a valence hole with $\frac{1}{3}$ of an electron with reversed spin (QE_R in the initial state) and $\frac{2}{3}$ of an electron with majority spin (two QH's in the final state), the emitted photon should be only partially polarized. This is in contrast to a completely polarized emission of the bright FCX complexes, $h\mathrm{QE}^*$ and $h\mathrm{QE}_2$. Therefore, the partially unpolarized emission in the "weak-coupling" regime ($d>\lambda$) could be an indication of the presence of QE_R's in the electron fluid.

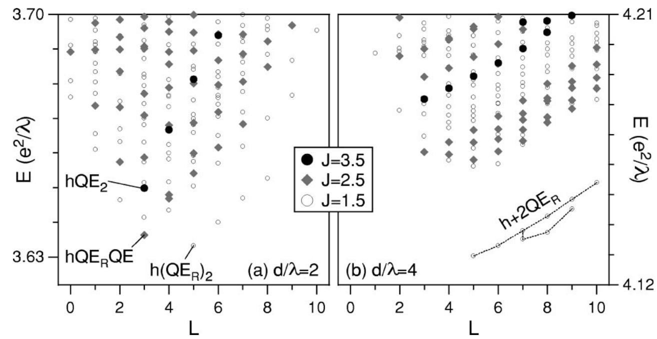

FIG. 6. The energy spectra (Coulomb energy E versus angular momentum L) of the system of $N=7$ electrons and one valence hole (h) on Haldane sphere at the monopole strength $2S=3(N-1)-2=16$, at the separations $d=2\lambda$ (a) and 4λ (b) between the electron and hole planes. Black dots, gray diamonds, and open circles mark states with the total electron spin $J=\frac{1}{2}N=\frac{7}{2}$ (maximum polarization), $J=\frac{1}{2}N-1=\frac{5}{2}$ (one reversed spin), and $J=\frac{1}{2}N-2=\frac{3}{2}$ (two reversed spins), respectively. The lowest-energy $J=\frac{7}{2}$, $\frac{5}{2}$, and $\frac{3}{2}$ states in (a) are the fractionally charged excitons, $h\mathrm{QE}_2$, $h\mathrm{QE}_R\mathrm{QE}$, and $h(\mathrm{QE}_R)_2$, respectively. The lowest-energy band of $J=\frac{3}{2}$ states marked with lines in (b) contains all possible states of two QE_R's and one h. λ is the magnetic length.

IV. CONCLUSION

Using exact numerical diagonalization, we have studied the low-energy spin-flip excitations of a 2DEG in the FQH regime (at $\nu=\frac{1}{3}$), so-called reversed-spin quasielectrons (QE_R's). The pseudopotentials $V(\mathcal{R})$ describing interaction of QE_R's with one another and with other Laughlin QP's have been calculated. From the form of the QE_R-QE_R pseudopotential it is shown that the Haldane-hierarchy $\nu=\frac{1}{3}$ daughter state of QE_R's formed in the parent $\nu=\frac{1}{3}$ Laughlin state of electrons is incompressible. This state corresponds to the total electron filling factor of $\nu=\frac{4}{11}$ and partial, 75% spin polarization. Because the analogous $\nu=\frac{1}{3}$ hierarchy state of QE's is known to be compressible, it is claimed that the experimentally observed[46] FQHE at $\nu=\frac{4}{11}$ confirms the formation of QE_R's and their Laughlin correlations in a 2DEG with low Zeeman splitting. Although the stability of mixed QE-QE_R hierarchy states is expected to be highly sensitive to the Zeeman energy E_Z, it is predicted that an incompressible [1,1,2] state that corresponds to $\nu=\frac{5}{13}$ and 80% spin polarization might form at appropriate E_Z. The interaction of QE_R's with a spatially separated valence-band hole has also been studied. In analogy to the so-called fractionally charged exciton (FCX) states $h\mathrm{QE}_n$, the spin-reversed complexes FCX_R that involve one or more QE_R's are predicted. Because QE and QE_R have different angular momenta, the optical selection rules for FCX and FCX_R are different, and, for example, $h\mathrm{QE}_R$ turns out radiative in contrast to the dark $h\mathrm{QE}$, while $h(\mathrm{QE}_R)_2$ is dark in contrast to the bright $h\mathrm{QE}_2$, Therefore, in addition to obvious difference in polarization, the emission from FCX and FCX_R states is expected to occur at a different energy and differently depend on temperature.

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