REVIEW ARTICLE

Composite fermions in fractional quantum Hall systems

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Abstract.   The composite-fermion (CF) picture offers a simple intuitive way of understanding many of the surprising properties of a strongly interacting two-dimensional electron fluid in a large magnetic field. The simple way in which the mean-field CF picture describes the low-lying bands of angular momentum multiplets for any value of the applied magnetic field is illustrated and compared with the results of exact numerical diagonalization of small systems. The reason for the success of the CF approach is discussed in some detail, and a CF hierarchy picture of the incompressible quantum fluid states is introduced. The CF picture is used to provide an understanding of the energy spectrum and photoluminescence of systems containing both electrons and valence band holes.

1. Introduction

The study of the electronic properties of quasi-two-dimensional (2D) systems has resulted in a number of remarkable discoveries in the past two decades [1–3]. Among the most interesting of these are the integral [4] and fractional [5] quantum Hall effects. In both of these effects, incompressible states of a 2D electron liquid are found at particular values of the electron density for a given value of the magnetic field applied normal to the 2D layer.

The integral quantum Hall effect (IQHE) is rather simple to understand. The incompressibility results from a cyclotron energy gap, $\hbar\omega_c$, in the single-particle spectrum. When all states below the gap are filled and all states above it are empty, it takes a finite energy $\hbar\omega_c$ to produce an infinitesimal compression. Excited states consist of electron–hole pair excitations and require a finite excitation energy. Both localized [6] and extended single-particle states are necessary to understand the experimentally observed behaviour of the magnetoconductivity [7].

The fractional quantum Hall effect (FQHE) is more difficult to understand and more interesting in terms of new basic physics. The energy gap that gives rise to the Laughlin [8] incompressible fluid state is completely the result of the interaction between the electrons. The elementary excitations are fractionally charged Laughlin quasiparticles, which satisfy fractional statistics [9]. The standard techniques of many-body perturbation theory are incapable of treating FQH systems because of the complete degeneracy of the single-particle levels in the absence of the interactions. Laughlin [8] was able to determine the form of the ground-state wavefunction and of the elementary excitations on the basis of physical insight into the nature of the many-body correlations. Striking confirmation of Laughlin's picture was obtained by exact diagonalization of the interaction Hamiltonian within the subspace of the lowest Landau level of small systems [10]. Jain [11], Lopez and Fradkin [12], and Halperin *et al* [13] have extended Laughlin's approach and developed a composite-fermion (CF) description of the 2D electron gas in a strong magnetic field. This CF description has offered a simple picture for the interpretation of many experimental results. However, the

underlying reason for the validity of many of the approximations used with the CF approach is not completely understood [14].

The object of this review is to present a simple and understandable summary of the CF picture as applied to FQH systems. Exact numerical calculations for up to eleven electrons on a spherical surface will be compared with the predictions of the mean-field CF picture. The CF hierarchy [15] will be introduced, and its predictions compared with numerical results. It will be shown that sometimes the mean-field CF hierarchy correctly predicts Laughlin-like incompressible ground states, and that sometimes it fails.

The CF hierarchy depends on the validity of the mean-field approximation. This seems to work well in predicting not only the Laughlin–Jain families of incompressible ground states at particular values of the applied magnetic field, but also in predicting the lowest-lying band of states at any value of the magnetic field. The question of when the mean-field CF picture works and why [14] will be discussed in some detail. As first suggested by Haldane [10], the behaviour of the pseudopotential $V(L)$ describing the energy of interaction of a pair of electrons as a function of their total angular momentum L is of critical importance. Some examples of other strongly interacting 2D fermion systems will be presented, and some problems not yet completely understood will be discussed.

The plan of the paper is as follows. In section 2 the single-particle states for electrons confined to a plane in the presence of an applied magnetic field [16] are introduced. The integral and fractional quantum Hall effects are discussed briefly. Haldane's idea [10] that the condensation of Laughlin quasiparticles leads to a hierarchy containing all odd-denominator fractions is discussed. In section 3 the numerical calculations for a finite number of electrons confined to a spherical surface in the presence of a radial magnetic field are discussed. Results for a ten-electron system at different values of the magnetic field are presented. In section 4 the ideas of fractional statistics and the Chern–Simons transformation are introduced. In section 5 Jain's CF approach [11] is outlined. The sequence of Jain condensed states (given by filling factor $\nu = n(1 + 2pn)^{-1}$, where n is any integer and p is a positive integer) is shown to result from the mean-field approximation. The application of the CF picture to electrons on a spherical surface is shown to predict the lowest band of angular momentum multiplets in a very simple way that involves only the elementary problem of addition of angular momenta [17]. In section 6 the two energy scales, the Landau-level separation $\hbar\omega_c$ and the Coulomb energy e^2/λ (where λ is the magnetic length), are discussed. It is emphasized that the Coulomb interactions and Chern–Simons gauge interactions between fluctuations (beyond the mean-field level) cannot possibly cancel for arbitrary values the applied magnetic field. The reason for the success of the CF picture is discussed in terms of the behaviour of the pseudopotential $V(L)$ and a kind of 'Hund's rule' for monopole harmonics [14]. In section 7, a phenomenological Fermi-liquid picture is introduced to describe low-lying excited states containing three or more Laughlin quasiparticles [18]. In section 8 the CF hierarchy picture [15] is introduced. Comparison with exact numerical results indicates that the behaviour of the quasiparticle pseudopotential is of critical importance in determining the validity of this picture at a particular level of the hierarchy. In section 9 systems containing electrons and valence band holes are investigated [19]. The photoluminescence and the role of excitons and negatively charged exciton complexes is discussed. The final section is a summary.

A book entitled '*Composite Fermions*' [1] edited by Heinonen has appeared quite recently. This book contains seven review articles covering different theoretical and experimental aspects of composite fermions, as well as numerous references to earlier review articles [2] and to original literature. The present review has a different scope and emphasis from these earlier reviews. The ones closest to the current review are those of Jain and Kamilla [1] and of Jain [2]. These papers introduce the CF model and propose wavefunctions describing fractional

filling by multiplying non-interacting wavefunctions for integral filling by a Jastrow–Laughlin correlation factor. After projecting onto the lowest Landau level, these wavefunctions are compared with the eigenfunctions obtained from exact diagonalization of small systems. For states belonging to the Laughlin–Jain sequence $\nu = n(1 + 2pn)^{-1}$ the overlap is very close to unity. A brief discussion of the Jain–Goldman CF hierarchy and of the operators D (which introduces Jastrow correlations), C (charge conjugation), and L (which increases the filling factor ν by unity) is given. The other theoretical reviews emphasize particular problems: the CF picture of the $\nu = 1/2$ state (Simon), edge states (Johnson and Kirczenow), the formal many-body field theory of the interaction beyond the mean-field level (Lopez and Fradkin), etc, none of which are considered in the present work.

The main emphasis of the present paper is on understanding why the CF mean-field picture works at all. This picture involves two completely different energy scales (Coulomb, $\propto \sqrt{B}$, and Chern–Simons, $\propto B$), and the CF interactions beyond the mean-field level cannot possibly cancel for arbitrary values of B as originally suggested [11]. The conclusions we reach make use of the ideas of fractional parentage from nuclear and atomic physics and depend critically on the interaction pseudopotential. Understanding how the pseudopotential affects the validity of the mean-field approximation's predictions allows us to investigate other related systems like electrons in higher Landau levels, charged fermion complexes (like the charged excitons), and Laughlin quasiparticles in a partially filled shell. A new and better understanding of the behaviour of fractional fillings of higher Landau levels (like $\nu = 7/3$ and 11/5), of the CF hierarchy, and of systems containing valence band holes as well as electrons results. This material, covered in sections 6 to 9, is unique to the current review.

2. Integral and fractional quantum Hall effects

The Hamiltonian for an electron confined to the x–y plane in the presence of a perpendicular magnetic field B is

$$H_0 = \frac{1}{2\mu}\left(p + \frac{e}{c}A\right)^2. \tag{1}$$

Here μ is the effective mass, $p = (p_x, p_y, 0)$ is the momentum operator, and $A(x, y)$ is the vector potential (whose curl gives B). For the 'symmetric gauge', $A = \frac{1}{2}B(y, -x, 0)$, the single-particle eigenfunctions [16] are of the form $\psi_{nm}(r, \theta) = e^{-im\theta} u_{nm}(r)$. The angular momentum of the state ψ_{nm} is $-m$ and its eigenenergy is given by

$$E_{nm} = \frac{1}{2}\hbar\omega_c(2n + 1 + |m| - m). \tag{2}$$

In these equations, $\omega_c = eB/\mu c$ is the cyclotron frequency, $n = 0, 1, 2, \ldots$, and $m = 0, \pm 1, \pm 2, \ldots$. The lowest energy states (lowest Landau level) have $n = 0$ and $m = 0, 1, 2, \ldots$ and energy $E_{0m} = \frac{1}{2}\hbar\omega_c$. It is convenient to introduce a complex coordinate $z = re^{-i\theta} = x - iy$, and to write the lowest-Landau-level wavefunctions as

$$\psi_{0m}(z) = N_m z^m e^{-|z|^2/4} \tag{3}$$

where N_m is a normalization constant. In this expression we have used the magnetic length $\lambda = \sqrt{\hbar c/eB}$ as the unit of length. The function $|\psi_{0m}|^2$ has its maximum value at a radius r_m which is proportional to $\sqrt{m}$. All single-particle states belonging to a given Landau level are degenerate, and separated in energy from neighbouring levels by $\hbar\omega_c$.

If the system has a 'finite radial range', then the m-values are restricted to being less than some maximum value ($m = 0, 1, 2, \ldots, N_\phi - 1$). The value of N_ϕ (the Landau-level degeneracy) is equal to the total flux through the sample, BC (where C is the area), divided

by the quantum of flux $\phi_0 = hc/e$. The filling factor ν is defined as the ratio of the number of electrons, N, to N_ϕ. When ν has an integral value, an infinitesimal decrease in the area C requires promotion of an electron across the cyclotron gap $\hbar\omega_c$ to the first unoccupied Landau level, making the system incompressible. This incompressibility together with the existence of both localized and extended states in the system is responsible for the observed behaviour of the magnetoconductivity of quantum Hall systems at integral filling factors [7].

In order to construct a many-electron wavefunction $\Psi(z_1, z_2, \ldots, z_N)$ corresponding to a completely filled lowest Landau level, the product function which places one electron in each of the $N_\phi = N$ orbitals ψ_{0m} ($m = 0, 1, \ldots, N_\phi - 1$) must be antisymmetrized. This can be done with the aid of a Slater determinant:

$$\Psi \propto \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ z_1^2 & z_2^2 & \cdots & z_N^2 \\ \vdots & \vdots & & \vdots \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1} \end{vmatrix} \exp\left(-\frac{1}{4}\sum_k |z_k|^2\right). \tag{4}$$

The determinant in equation (4) is the well-known Vandemonde determinant. It is not difficult to show that it is equal to $\prod_{i<j}(z_i - z_j)$. Of course, N_ϕ is equal to N (since each of the N_ϕ orbitals is occupied by one electron) and the filling factor $\nu = 1$.

Laughlin noticed that if the factor $(z_i - z_j)$ arising from the Vandemonde determinant was replaced by $(z_i - z_j)^{2p+1}$, where p was an integer, the wavefunction

$$\Psi_{2p+1} \propto \prod_{i<j}(z_i - z_j)^{2p+1} \exp\left(-\frac{1}{4}\sum_i |z_i|^2\right) \tag{5}$$

would be antisymmetric, keep the electrons further apart (and therefore reduce the Coulomb repulsion), and correspond to a filling factor $\nu = (2p + 1)^{-1}$. This results because the highest power of z_i in the polynomial factor in Ψ_{2p+1} is $(2p + 1)(N - 1)$ and it must be equal to the highest orbital index ($m = N_\phi - 1$), giving $N_\phi - 1 = (2p + 1)(N - 1)$ and $\nu = N/N_\phi$ equal to $(2p + 1)^{-1}$ in the limit of large systems. The additional factor $\prod_{i<j}(z_i - z_j)^{2p}$ multiplying $\Psi_{m=1}$ is the Jastrow factor which accounts for correlations between electrons.

It is observed experimentally that states with filling factors $\nu = 2/5, 3/5, 3/7$, etc exhibit FQH behaviour in addition to the Laughlin $\nu = (2p + 1)^{-1}$ states. Haldane [10] suggested that a hierarchy of condensed states arose from the condensation of quasiparticles (QPs) of 'parent' FQH states. In his picture, Laughlin condensed states of the electron system occurred when $N_\phi = (2p + 1)N_e$, where the exponent $2p + 1$ in equation (5) was an odd integer and the symbol N_e denoted the number of electrons. Condensed QP states occurred when $N_e = 2qN_{QP}$, because the number of places available for inserting a QP in a Laughlin state was N_e. Haldane required the exponent $2q$ to be even 'because the QPs are bosons'. This scheme gives rise to a hierarchy of condensed states which contains all odd-denominator fractions. Haldane cautioned that the validity of the hierarchy scheme at a particular level depended upon the QP interactions which were totally unknown.

3. Numerical study of small systems

Haldane [10] introduced the idea of putting a small number of electrons on a spherical surface of radius R at the centre of which is a magnetic monopole of strength $2S\phi_0$. The single-particle Hamiltonian can be expressed as [20]

$$H_0 = \frac{\hbar^2}{2\mu R^2}(\mathbf{L} - S\hat{R})^2 \tag{6}$$

where L is the angular momentum operator (in units of $\hbar$), $\hat{R}$ is the unit vector in the radial direction, and μ is the mass. The components of L satisfy the usual commutation rules $[L_\alpha, L_\beta] = i\epsilon_{\alpha\beta\gamma} L_\gamma$. The eigenstates of H_0 can be denoted by $|l, m\rangle$; they are eigenfunctions of L^2 and L_z with eigenvalues $l(l+1)$ and m, respectively. The lowest energy eigenvalue (shell) occurs for $l = S$ and has energy $\frac{1}{2}\hbar\omega_c$. The nth excited shell has $l = S + n$, and

$$E_n = \frac{\hbar\omega_c}{2S}\left[l(l+1) - S^2\right] = \hbar\omega_c\left[n + \frac{1}{2} + \frac{n(n+1)}{2S}\right] \tag{7}$$

where the cyclotron energy is equal to $\hbar\omega_c = S\hbar^2/\mu R^2$ and the magnetic length is $\lambda = R/\sqrt{S}$. If we concentrate on a partially filled lowest Landau level we have only $N_\phi = 2S+1$ degenerate single-particle states (since the electron angular momentum l must be equal to S and its z-component m can take on values between $-l$ and l). The Hilbert space $\mathcal{H}_{\mathrm{MB}}$ of N electrons in these N_ϕ single-particle states contains $N_{\mathrm{MB}} = N_\phi![N!(N_\phi - N)!]^{-1}$ antisymmetric many-body states. The single-particle configurations $|m_1, m_2, \ldots, m_N\rangle = c^\dagger_{m_1} c^\dagger_{m_2} \cdots c^\dagger_{m_N}|\mathrm{vac}\rangle$ can be chosen as a basis of $\mathcal{H}_{\mathrm{MB}}$. Here $c^\dagger_m$ creates an electron in the single-particle state $|l = S, m\rangle$, and $|\mathrm{vac}\rangle$ is the vacuum state. The space $\mathcal{H}_{\mathrm{MB}}$ can also be spanned by the angular momentum eigenfunctions, $|L, M, \alpha\rangle$, where L is the total angular momentum, M its z-component, and α is a label which distinguishes different multiplets with the same L. If $\hbar\omega_c \gg e^2/\lambda$, the diagonalization of the interaction Hamiltonian

$$H_I = \sum_{i<j} \frac{e^2}{r_{ij}} \tag{8}$$

in the Hilbert space $\mathcal{H}_{\mathrm{MB}}$ of the lowest Landau level gives an excellent approximation to exact eigenstates of an interacting N-electron system. The single-particle configuration basis is particularly convenient since the many-body interaction matrix elements in this basis, $\langle m_1, m_2, \ldots, m_N|H_I|m'_1, m'_2, \ldots, m'_N\rangle$, are expressed through the two-body ones, $\langle m_1, m_2|H_I|m'_1, m'_2\rangle$, in a very simple way. On the other hand, using the angular momentum eigenstates $|L, M, \alpha\rangle$ allows the explicit decomposition of the total Hilbert space $\mathcal{H}_{\mathrm{MB}}$ into total-angular-momentum eigensubspaces. Because the interaction Hamiltonian is a scalar, the Wigner–Eckart theorem tells us that

$$\langle L', M', \alpha'|H_I|L, M, \alpha\rangle = \delta_{LL'}\delta_{MM'} V_{\alpha\alpha'}(L) \tag{9}$$

where the reduced matrix element

$$V_{\alpha\alpha'}(L) = \langle L, \alpha'|H_I|L, \alpha\rangle \tag{10}$$

is independent of M. The eigenfunctions of L are simpler to find than those of H_I, because efficient numerical techniques exist for obtaining eigenfunctions of operators with known eigenvalues. Finding the eigenfunctions of L and then using the Wigner–Eckart theorem considerably reduces the dimensions of the matrices that must be diagonalized to obtain eigenvalues of H_I. Some matrix dimensions are listed in table 1, where the degeneracy of the lowest Landau level and the dimensions of the total many-body Hilbert space, N_{MB}, and of the largest M-subspace, $N_{\mathrm{MB}}(M = 0)$, are given for the Laughlin $\nu = 1/3$ state of systems of six to eleven electrons (the N-electron Laughlin $\nu = (2p + 1)^{-1}$ state occurs at $N_\phi = (2p + 1)(N - 1)$). For example, in the eleven-electron system at $\nu = 1/3$, the $L = 0$ block that must be diagonalized to obtain the Laughlin ground state is only 1160 by 1160, small compared to the total dimension of 1 371 535 for the entire $M = 0$ subspace.

Typical results for the energy spectrum are shown in figure 1 for $N = 10$ and a few different values of $2S$ between 21 and 30. The low-energy bands marked with open circles and solid lines will be discussed in detail in the following sections. Frames (a) and (f) show two $L = 0$ incompressible ground states: the Laughlin state at $\nu = 1/3$ and the Jain state at $\nu = 2/5$, respectively. In other frames, a number of QPs form the lowest energy bands.

Table 1. The Landau-level degeneracy $N_\phi = 2S + 1$ and the dimensions of the total N-electron Hilbert space, N_{MB}, and of the largest M-subspace, $N_{MB}(M = 0)$, at the filling factor $\nu = 1/3$.

N	N_ϕ	N_{MB}	$N_{MB}(M = 0)$
6	16	8008	338
7	19	50388	1656
8	22	319770	8512
9	25	2042975	45207
10	28	13123110	246448
11	31	84672315	1371535

4. Chern–Simons transformation and statistics in 2D systems

Before discussing the Chern–Simons gauge transformation and its relation to particle statistics, it is useful to look at a system of two particles each of charge $-e$ and mass μ, confined to a plane, in the presence of a perpendicular magnetic field $\boldsymbol{B} = (0, 0, B) = \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r})$. Because $\boldsymbol{A}$ is linear in the coordinate $\boldsymbol{r} = (x, y)$ (e.g., in the symmetric gauge, $\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{2}B(y, -x)$), the Hamiltonian separates into the centre-of-mass (CM) and relative (REL) coordinate pieces, with $\boldsymbol{R} = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2)$ and $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ being the CM and REL coordinates, respectively. The energy spectra of H_{CM} and H_{REL} are identical to that of a single particle of mass μ and charge $-e$. We have already seen that for the lowest Landau level $\psi_{0m} = N_m r^m \mathrm{e}^{-im\phi} \mathrm{e}^{-r^2/4\lambda^2}$. For the relative motion ϕ is equal to $\phi_1 - \phi_2$, and the interchange of the pair, $P\psi(\boldsymbol{r}_1, \boldsymbol{r}_2) = \psi(\boldsymbol{r}_2, \boldsymbol{r}_1)$, is accomplished by replacing ϕ by $\phi + \pi$. In 3D systems, where two consecutive interchanges must result in the original wavefunction, this implies that $\mathrm{e}^{im\pi}$ must be equal to either $+1$ (m even; bosons) or -1 (m odd; fermions). It is well known [21] that for 2D systems m need not be an integer. Interchange of a pair of identical particles can give $P\psi(\boldsymbol{r}_1, \boldsymbol{r}_2) = \mathrm{e}^{i\theta\pi}\psi(\boldsymbol{r}_1, \boldsymbol{r}_2)$, where the statistical parameter θ can assume non-integral values leading to anyon statistics.

A Chern–Simons (CS) transformation is a singular gauge transformation [13] in which an electron creation operator $\psi_e^\dagger(\boldsymbol{r})$ is replaced by a composite particle operator $\psi^\dagger(\boldsymbol{r})$ given by

$$\psi^\dagger(\boldsymbol{r}) = \psi_e^\dagger(\boldsymbol{r}) \exp\left[i\alpha \int \mathrm{d}^2 r' \, \arg(\boldsymbol{r} - \boldsymbol{r}')\psi^\dagger(\boldsymbol{r}')\psi(\boldsymbol{r}')\right]. \tag{11}$$

Here $\arg(\boldsymbol{r} - \boldsymbol{r}')$ is the angle that the vector $\boldsymbol{r} - \boldsymbol{r}'$ makes with the x-axis and α is an arbitrary parameter. The kinetic energy operator can be written in terms of the transformed operator as

$$K = \frac{1}{2\mu} \int \mathrm{d}^2 r \, \psi^\dagger(\boldsymbol{r}) \left[-i\hbar\boldsymbol{\nabla} + \frac{e}{c}\boldsymbol{A}(\boldsymbol{r}) + \frac{e}{c}\boldsymbol{a}(\boldsymbol{r})\right]^2 \psi(\boldsymbol{r}). \tag{12}$$

Here

$$\boldsymbol{a}_{r'}(\boldsymbol{r}) = \frac{\alpha\phi_0}{2\pi} \frac{\hat{z} \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^2} \tag{13}$$

and

$$\boldsymbol{a}(\boldsymbol{r}) = \alpha\phi_0 \int \mathrm{d}^2 r' \, \boldsymbol{a}_{r'}(\boldsymbol{r})\psi^\dagger(\boldsymbol{r}')\psi(\boldsymbol{r}') \tag{14}$$

where $\hat{z}$ is a unit vector perpendicular to the 2D layer. The CS transformation can be thought of as an attachment to each particle of flux tube carrying a fictitious flux $\alpha\phi_0$ (where $\phi_0 = hc/e$ is the quantum of flux) and a fictitious charge $-e$ which couples in the standard way to the vector potential caused by the flux tubes on every other particle. The $\boldsymbol{a}_{r'}(\boldsymbol{r})$ is interpreted as the vector potential at position $\boldsymbol{r}$ due to a magnetic flux of strength $\alpha\phi_0$ localized at $\boldsymbol{r}'$, and
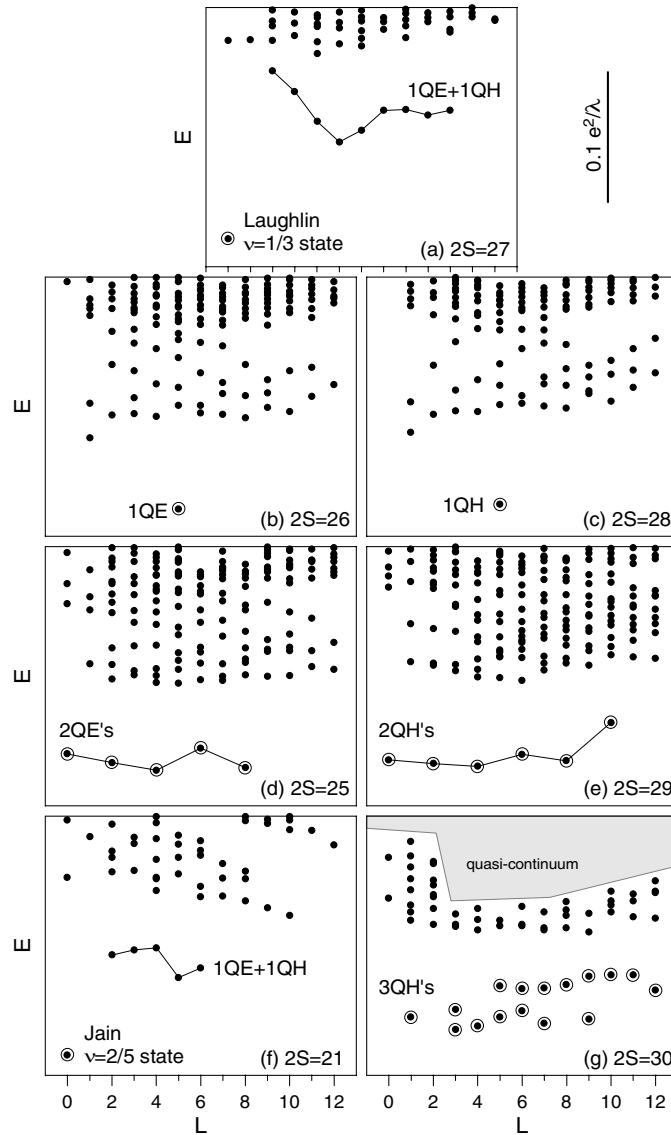
Figure 1. The energy spectra of ten electrons in the lowest Landau level calculated on a Haldane sphere with $2S$ between 21 and 30. The open circles and solid lines mark the lowest energy bands with the fewest composite-fermion quasiparticles.

$a(r)$ is the total vector potential at position r due to all CS fluxes. The CS magnetic field associated with the particle at r' is $b(r) = \nabla \times a_{r'}(r) = \alpha \phi_0 \delta(r - r')\hat{z}$. Because two charged particles cannot occupy the same position, one particle never senses the magnetic field of other particles, but it does sense the vector potential resulting from their CS fluxes. The classical equations of motion are unchanged by the presence of the CS flux, but the quantum statistics of the particles are changed unless α is an even integer.

For the two-particle system, the vector potential associated with the CS flux $a_{r_2}(r_1)$ depends only on the relative coordinate $r = r_1 - r_2$. When $a(r)$ is added to $A(r)$, the vector

potential of the applied magnetic field, the Schrödinger equation has a solution

$$\tilde{\psi}_m = \mathrm{e}^{-\mathrm{i}\alpha\phi} \psi_m \tag{15}$$

where ψ_m is the solution with $\alpha = 0$ (i.e. in the absence of CS flux). If α is an odd integer, boson and fermion statistics are interchanged; if α is even, no change in statistics occurs and electrons are transformed into composite fermions with an identical energy spectrum.

The Hamiltonian for the composite particle system (charged particles with attached flux tubes) is much more complicated than the original system with $\alpha = 0$. What is gained by making the CS transformation? The answer is that one can use the 'mean-field' approximation in which $\boldsymbol{A}(\boldsymbol{r}) + \boldsymbol{a}(\boldsymbol{r})$, the vector potential of the external plus CS magnetic fields, is replaced by $\boldsymbol{A}(\boldsymbol{r}) + \langle \boldsymbol{a}(\boldsymbol{r}) \rangle$, where $\langle \boldsymbol{a}(\boldsymbol{r}) \rangle$ is the mean-field value of $\boldsymbol{a}(\boldsymbol{r})$ obtained by simply replacing $\varrho(\boldsymbol{r}') = \psi^\dagger(\boldsymbol{r}')\psi(\boldsymbol{r}')$ by its average value ϱ_0 in equation (14). A mean-field energy spectrum can be constructed in which the massive degeneracy of the original partially filled electron Landau level disappears. One might then hope to treat both the Coulomb interaction and the CS gauge field interactions among the fluctuations (beyond the mean-field level) by standard many-body perturbation techniques (e.g. by the random-phase approximation, RPA). Unfortunately, there is no small parameter for a many-body perturbation expansion unless α, the number of CS flux quanta attached to each particle, is small compared to unity. However, a Landau–Silin-type [22] Fermi-liquid approach can take account of the short-range correlations phenomenologically. A number of excellent papers on anyon superconductivity [23] treat CS gauge interactions by standard many-body techniques. Halperin and collaborators [13] have treated the half-filled Landau level as a liquid of composite fermions moving in zero effective magnetic field. Their RPA–Fermi-liquid approach gives a surprisingly satisfactory account of the properties of that state.

The vector potential associated with fluctuations beyond the mean-field level is given by $\delta \boldsymbol{a}(\boldsymbol{r}) = \boldsymbol{a}(\boldsymbol{r}) - \langle \boldsymbol{a}(\boldsymbol{r}) \rangle$. The perturbation to the mean-field Hamiltonian contains both linear and quadratic terms in $\delta \boldsymbol{a}(\boldsymbol{r})$, resulting in both two-body—containing $\varrho(\boldsymbol{r}_1)\varrho(\boldsymbol{r}_2)$—and three-body—containing $\varrho(\boldsymbol{r}_1)\varrho(\boldsymbol{r}_2)\varrho(\boldsymbol{r}_3)$—interaction terms. The three-body interaction terms are usually ignored, though for α of the order of unity this approximation is of questionable validity.

5. Jain's composite-fermion picture

Jain noted that in the mean-field approximation, an effective filling factor ν^* of the composite fermions was related to the electron filling factor ν by the relation

$$(\nu^*)^{-1} = \nu^{-1} - 2p. \tag{16}$$

Remember that ν^{-1} is equal to the number of flux quanta of the applied magnetic field per electron, and $2p$ is the (even) number of CS flux quanta (oriented opposite to the applied magnetic field) attached to each electron in the CS transformation. Equation (16) implies that when $\nu^* = \pm 1, \pm 2, \ldots$ (negative values correspond to the effective magnetic field $\boldsymbol{B}^*$ seen by the CFs oriented opposite to $\boldsymbol{B}$) and a non-degenerate mean-field CF ground state occurs, then $\nu = \nu^*(1 + 2p\nu^*)^{-1}$. This Jain sequence of condensed states ($\nu = 1/3, 2/5, 3/7, \ldots$ and $\nu = 2/3, 3/5, \ldots$ for $p = 1$) is the set of FQH states most prominent in experiment. When ν^* is not an integer, QPs of the neighbouring Jain state will occur.

It is quite remarkable that the mean-field CF picture predicts not only the Jain sequence of incompressible ground states, but also the correct band of low-energy states for any value of the applied magnetic field. This is very nicely illustrated for the case of N electrons on a Haldane sphere. When the monopole strength seen by an electron has the value $2S$, the effective monopole strength seen by a CF is $2S^* = 2S - 2p(N-1)$. This equation reflects the

fact that a given CF senses the vector potential produced by the CS flux on all other particles, but not its own CS flux. In table 2 the ten-particle system is described for a number of values of $2S$ between 29 and 15. The Laughlin $\nu = 1/3$ state occurs at $2S_3 = 3(N - 1) = 27$. For values of $2S$ different from this value, $2S - 2S_3 = \pm N_{QP}$ ('+' corresponds to quasiholes, QH, and '−' to quasielectrons, QE). Let us apply the CF description to the ten-electron spectra in figure 1. At $2S = 27$, we take $p = 1$ and attach two CS flux quanta to each electron. This gives $2S^* = 9$, so the ten CFs completely fill the $2S^* + 1$ states in the lowest angular momentum shell (lowest Landau level). There is a gap $\hbar\omega_c^* = \hbar eB^*/\mu c$ to the next shell, which is responsible for the incompressibility of the Laughlin state. Just as $|S|$ played the role of the angular momentum of the lowest shell of electrons, $l^* = |S^*|$ plays the role of the CF angular momentum and $2|S^*| + 1$ is the degeneracy of the CF shell. Thus, the states with $2S = 26$ and 28 contain a single quasielectron (QE) and quasihole (QH), respectively. For the QE state, $2S^* = 8$, and the lowest shell of angular momentum $l_0^* = 4$ can accommodate only nine CFs. The tenth is the QE in the $l_1^* = l_0^* + 1 = 5$ shell, giving the total angular momentum $L = 5$. For the QH state, $2S^* = 10$, and the lowest shell can accommodate eleven CFs each with angular momentum $l_0^* = 5$. The one empty state (QH) gives $L = l^* = 5$. For $2S = 25$ we obtain $2S^* = 7$, and there are two QEs each of angular momentum $l_1^* = 9/2$ in the first excited CF shell. Adding the angular momenta of the two QEs gives the band of multiplets $L = 0, 2, 4, 6$, and 8. Similarly, for $2S = 29$ we obtain $2S^* = 11$, and there are two QHs each with $l_0^* = 11/2$, resulting in the allowed pair states at $L = 0, 2, 4, 6, 8$, and 10. At $2S = 21$, the lowest shell with $l_0^* = 3/2$ can accommodate only four CFs, but the other six CFs exactly fill the excited $l_1^* = 5/2$ shell. The resulting incompressible ground state is the Jain $\nu = 2/5$ state, since $\nu^* = 2$ for the two filled shells. A similar argument leads to $\nu^* = -2$ (the minus sign means B^* oriented opposite to B) and $\nu = 2/3$ at $2S = 15$. At $2S = 30$, the addition of three QH angular momenta of $l_0^* = 6$ gives the following band of low-lying multiplets: $L = 1$, $3^2, 4, 5^2, 6^2, 7^2, 8, 9^2, 10, 11, 12, 13$, and 15. As demonstrated on an example in figure 1, this simple mean-field CF picture correctly predicts the band of low-energy multiplets for any number of electrons N and for any value of $2S$.

Table 2. The effective CF monopole strength $2S^*$, the numbers of CF quasiparticles (quasielectrons, N_{QE}, and quasiholes, N_{QH}), the angular momentum of the lowest CF shell l^*, the CF and electron filling factors ν^* and ν, and the angular momenta L of the lowest-lying band of multiplets for a ten-electron system at $2S$ between 29 and 15.

$2S$	29		28	27	26	25		21	15
$2S^*$	11		10	9	8	7		3	−3
N_{QH}	2		1	0	0	0		0	0
N_{QE}	0		0	0	1	2		6	6
l_0^*	11/2		5	9/2	4	7/2		3/2	3/2
ν^*				1				2	−2
ν				1/3				2/5	2/3
L	0, 2, 4, 6, 8, 10		5	0	5	0, 2, 4, 6, 8		0	0

6. Energy scales and the electron pseudopotentials

The mean-field composite-fermion picture is remarkably successful in predicting the low-energy multiplets in the spectrum of N electrons on a Haldane sphere. It was suggested originally that this success resulted from the cancellation of the Coulomb and Chern–Simons

gauge interactions among fluctuations beyond the mean-field level. In figure 2, we show the lowest bands of multiplets for eight non-interacting electrons and for the same number of non-interacting mean-field CFs at $2S = 21$. The energy scale associated with the CS gauge interactions which convert the electron system in frame (a) to the CF system in frame (b) is $\hbar\omega_c^* \propto B$. The energy scale associated with the electron–electron Coulomb interaction is $e^2/\lambda \propto \sqrt{B}$. The Coulomb interaction lifts the degeneracy of the non-interacting electron bands in frame (a). However, for very large value of B the Coulomb energy can be made arbitrarily small compared to the CS energy (as marked with a shaded rectangle in figure 2), i.e. to the separation between the CF Landau levels. The energy separations in the mean-field CF model are completely wrong, but the structure of the low-lying states (i.e., which angular momentum multiplets form the low-lying bands) is very similar to that of the fully interacting electron system and completely different from that of the non-interacting electron system.
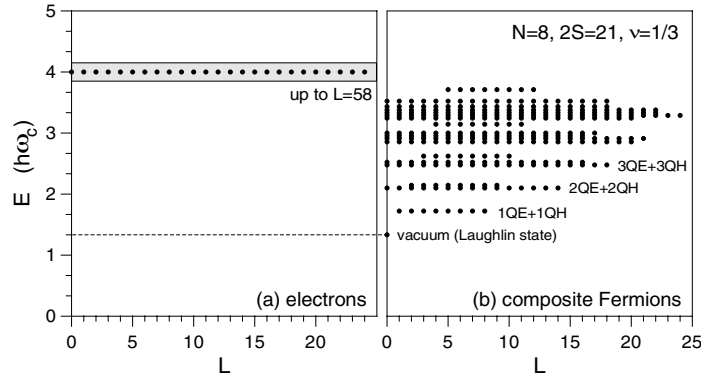

Figure 2. The energy spectra of eight (a) non-interacting electrons and (b) non-interacting composite fermions. The characteristic energy of the Coulomb interaction is marked in frame (a) with a shaded rectangle.

6.1. Two-fermion problem

An intuitive picture of why this occurs can be obtained by considering the two-fermion problem. The relative (REL) motion of a pair of electrons (ij) is described by a coordinate $z_{ij} = z_i - z_j = r_{ij}e^{-i\phi_{ij}}$, and for the lowest Landau level its wavefunction contains a factor z_{ij}^m, where $m = 1, 3, 5, \ldots$. If every pair of particles has identical behaviour, the many-particle wavefunction must contain a similar factor for each pair giving a total factor $\prod_{i<j} z_{ij}^m$. As we have seen, the highest power of z_i in this product is $m(N-1)$. If $m(N-1)$ is equal to $N_\phi - 1 = 2S$, the maximum value of the z-component of the single-particle angular momentum, the Laughlin $\nu = m^{-1}$ wavefunction results. For electrons, the mth cyclotron orbit, whose radius is r_m, encloses a flux $m\phi_0$ (i.e. $\pi r_m^2 B = m\phi_0$). For a Laughlin $\nu = m^{-1}$ state the pair function must have a radius $r_m = r_1\sqrt{m}$. Let us describe the CF orbits by radius $\varrho_{\tilde{m}}$ and require that the $\tilde{m}$th orbit enclose $\tilde{m}$ flux quanta. It is apparent that if a flux tube carrying two flux quanta (oriented opposite to the applied magnetic field B) is attached to each electron in the CS transformation of the $\nu = 1/3$ state, the smallest orbit of radius $\varrho_{\tilde{m}=1}$ has exactly the same size as $r_{m=3}$. Both orbits enclose three flux quanta of the applied field, but the CF orbit also encloses the two oppositely oriented CS flux quanta attached to the electrons to form the CFs. In the absence of electron–electron interactions, the energies of these orbits are unchanged, since they still belong to the degenerate single-particle states of the lowest Landau level.

In the mean-field approximation the CS fluxes are replaced by a spatially uniform magnetic field, leading to an effective field $B^* = B/m$. The orbits for the CF pair states in the mean-field approximation are exactly the same as those of the exact CS Hamiltonian. The smallest orbit has radius $\varrho_{\tilde{m}=1}$ equivalent to the electron orbit $r_{m=3}$. However, in the mean-field approximation, the energies are changed (because $\omega_c^* = eB^*/\mu c$ replaces ω_c). This energy change leads to completely incorrect mean-field CF energies, but the mean-field CF orbitals give the correct structure to the low-lying set of multiplets.

In the presence of a repulsive interaction, the low-lying energy states will have the largest possible value of m. For a monopole strength $2S = m(N - 1)$, where m is an odd integer, every pair can have radius r_m and avoid the large repulsion associated with $r_1, r_3, \ldots, r_{m-2}$. These ideas can be made somewhat more rigorous by using methods of atomic and nuclear physics for studying angular momentum shells of interacting fermions.

6.2. Two-body interaction pseudopotential

As first suggested by Haldane [10], the behaviour of the interacting many-electron system depends entirely on the behaviour of the two-body interaction pseudopotential, which is defined as the interaction energy V of a pair of electrons as a function of their pair angular momentum. In the spherical geometry, in order to allow for meaningful comparison of the pseudopotentials obtained for different values of $2S$ (and thus different single-electron angular momenta l), it is convenient to use the 'relative' angular momentum $\mathcal{R} = 2l - L_{12}$ rather than L_{12} (the length of $\hat{L}_{12} = \hat{l}_1 + \hat{l}_2$). The pair states with a given $\mathcal{R} = m$ (an odd integer) obtained on a sphere for different $2S$ are equivalent and correspond to the pair state on a plane with the relative (REL) motion described by angular momentum m and radius r_m. The pair state with the smallest allowed orbit (and largest repulsion) has $\mathcal{R} = 1$ on a sphere or $m = 1$ on a plane, and larger $\mathcal{R}$ and m means larger average separation. In the limit of $\lambda/R \to 0$ (i.e., either $2S \to \infty$ or $R \to \infty$), the pair wavefunctions and energies calculated on a sphere for $\mathcal{R} = m$ converge to the planar ones (ψ_{0m} and its energy).

The pseudopotentials $V(\mathcal{R})$ are plotted in figure 3 for a number of values of the monopole strength $2S$. The open circles mark the pseudopotential calculated on a plane ($\mathcal{R} = m$). At small $\mathcal{R}$ the pseudopotentials rise very quickly with decreasing $\mathcal{R}$ (i.e. separation). More importantly, they increase more quickly than linearly as a function of $L_{12}(L_{12} + 1)$. The pseudopotentials with this property form a class of so-called 'short-range' repulsive pseudopotentials [14]. If the repulsive interaction has short range, the low-energy many-body states must, to the extent that it is possible, avoid pair states with the smallest values of $\mathcal{R}$ (or m) and the maximum two-body repulsion.

6.3. Fractional grandparentage

It is well known in atomic and nuclear physics that the eigenfunction of an N-fermion system of total angular momentum L can be written as

$$\left| l^N, L\alpha \right\rangle = \sum_{L_{12}} \sum_{L'\alpha'} G_{L\alpha, L'\alpha'}(L_{12}) \left| l^2, L_{12}; l^{N-2}, L'\alpha'; L \right\rangle. \tag{17}$$

Here, the totally antisymmetric state $\left| l^N, L\alpha \right\rangle$ is expanded in the basis of states $\left| l^2, L_{12}; l^{N-2}, L'\alpha'; L \right\rangle$ which are antisymmetric under permutation of particles 1 and 2 (which are in the pair eigenstate of angular momentum L_{12}) and under permutation of particles $3, 4, \ldots, N$ (which are in the $(N - 2)$-particle eigenstate of angular momentum L'). The labels α (and α') distinguish independent states with the same angular momentum L (and L').
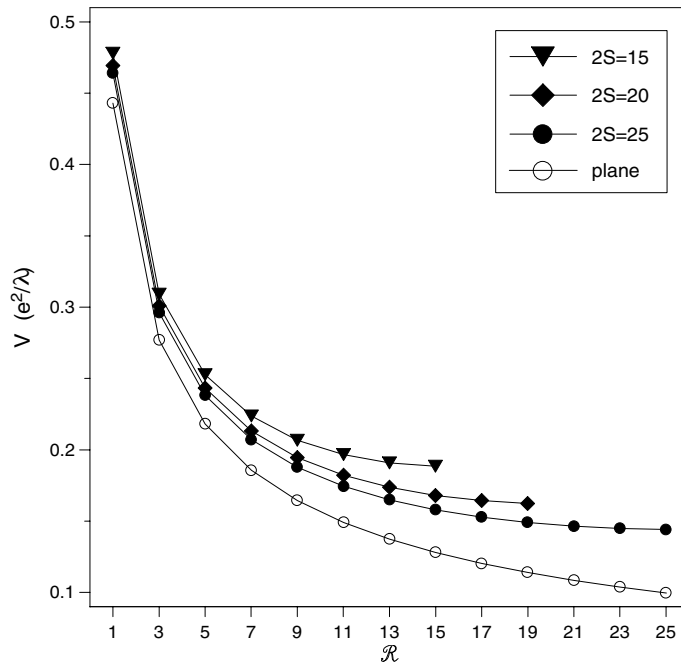
Figure 3. The pseudopotentials of the Coulomb interaction in the lowest Landau level calculated on a Haldane sphere with $2S = 15, 20$, and 25 (solid triangles, diamonds, and circles, respectively), and on a plane (open circles).

The expansion coefficient $G_{L\alpha,L'\alpha'}(L_{12})$ is called the coefficient of fractional grandparentage (CFGP).

For a simple three-fermion system, equation (17) reduces to

$$\left| l^3, L\alpha \right\rangle = \sum_{L_{12}} F_{L\alpha}(L_{12}) \left| l^2, L_{12}; l; L \right\rangle \tag{18}$$

and $F_{L\alpha}(L_{12})$ is called the coefficient of fractional parentage (CFP). In the lowest Landau level, the individual fermion angular momentum l is equal to S, half the monopole strength, and the number of independent multiplets of angular momentum L that can be formed by addition of the angular momenta of three identical fermions is given in table 3.

Low-energy many-body states must, to the extent it is possible, avoid parentage from pair states with the largest repulsion (pair states with maximum angular momenta L_{ij} or minimum $\mathcal{R}$). In particular, we expect that the lowest energy multiplets will avoid parentage from the pair state with $\mathcal{R} = 1$. If $\mathcal{R} = 1$, i.e. $L_{12} = 2l - 1$, the smallest possible value of the total angular momentum L of the three-fermion system is obtained by addition of vectors $\boldsymbol{L}_{12}$ (of length $2l - 1$) and $\boldsymbol{l}_3$ (of length l), and it is equal to $|(2l - 1) - l| = l - 1$. Therefore, the three-particle states with $L < l - 1$ must not have parentage from $\mathcal{R} = 1$. It is straightforward to show that if $L < l - (2p - 1)$, where $p = 1, 2, 3, \ldots$, the three-electron multiplet at L has no fractional parentage from $\mathcal{R} \leqslant 2p - 1$. The multiplets that must avoid one, two, or three smallest values of $\mathcal{R}$ are underlined with an appropriate number of lines in table 3 and listed in table 4. This gives the results in table 4, the values of $2L$ that avoid $\mathcal{R} = 1, 3$, and 5 for various values of $2l$. The $L = 0$ states that appear at $2l = 6$ ($\mathcal{R} \geqslant 3$), $2l = 10$ ($\mathcal{R} \geqslant 5$), and $2l = 14$ ($\mathcal{R} \geqslant 7$) are the only states for these values of $2l$ that can avoid one, two, or three largest pseudopotential parameters, respectively, and therefore are the non-degenerate ($L = 0$)

Table 3. The number of times an L-multiplet appears for a system of three electrons of angular momentum l. Top: even values of $2l$; bottom: odd values of $2l$. Blank spaces are equivalent to zeros.

$2l$	$2S=0$	2	4	6	8	10	12	14	16	18	20	22	24	26
2	1													
4		1		1										
6	1		1	1	1		1							
8		1		2	1	1	1	1		1				
10	1		1	1	2	1	2	1	1	1	1		1	
12		1		2	1	2	2	2	1	2	1	1	1	1
14	1		1	1	2	1	3	2	2	2	2	1	2	1

$2l$	$2S=1$	3	5	7	9	11	13	15	17	19	21	23	25	27
3		1												
5		1	1		1									
7		1	1	1	1	1		1						
9		1	1	1	2	1	1	1	1		1			
11		1	1	1	2	2	1	2	1	1	1	1		1
13		1	1	1	2	2	2	2	2	1	2	1	1	1

Table 4. The allowed values of $2L$ for a three-electron system that must have $\mathcal{R} \geqslant 3$, $\mathcal{R} \geqslant 5$, and $\mathcal{R} \geqslant 7$. The listed $2L$-values correspond to the underlined L-multiplets in table 3.

$2l$	$2L\ (\mathcal{R} \geqslant 3)$	$2L\ (\mathcal{R} \geqslant 5)$	$2L\ (\mathcal{R} \geqslant 7)$
6	0		
7	3		
8	2		
9	3, 5		
10	0, 4, 6	0	
11	3, 5, 7	3	
12	$2, 6^2, 8$	2	
13	$3, 5, 7, 9^2$	3, 5	
14	$0, 4, 6, 8^2, 10$	0, 4, 6	0

ground states. They are the Laughlin $\nu = 1/3$, 1/5, and 1/7 states.

If only a single multiplet belongs to an angular momentum subspace, its form is completely determined by the requirement that it is an eigenstate of angular momentum with a given eigenvalue L. The wavefunction and the type of many-body correlations do not depend on the form of the interaction pseudopotential. For interactions that do not have short range, the state that avoids the largest two-body repulsion (e.g. the $L = 0$ multiplet at $2l = 6$) might not have the lowest total three-body interaction energy and be the ground state. If more than one multiplet belongs to a given angular momentum eigenvalue (e.g., two multiplets occur at $L = 3$ for $2l = 8$), the interparticle interaction must be diagonalized in this subspace (two dimensional for $2l = 8$ and $L = 3$). Whether the lowest energy eigenstate in this subspace has Laughlin-type correlations, i.e. avoids as much as possible the largest two-body repulsion, depends critically on the short range of the interaction pseudopotential. For the Coulomb interaction, we find that the Laughlin correlations occur and, whenever possible, the CFP of the lowest-lying multiplets virtually vanishes (it would vanish exactly for an 'ideal' short-range pseudopotential which increases infinitely quickly with decreasing $\mathcal{R}$). For example, for the lower energy eigenstate at $L = 3$ and $2l = 8$, the CFP for $\mathcal{R} = 1$ is less than 10^{-3}. A similar

thing occurs at $2S = 9$ for $L = 9/2$, at $2S = 10$ for $L = 4$ and 6, at $2S = 11$ for $L = 9/2$, 11/2, and 15/2, at $2S = 12$ for $L = 5, 6, 7$, and 9, at $2S = 13$ for $L = 11/2, 13/2, 15/2$, 17/2, and 21/2, and at $2S = 14$ for $L = 6^2, 7, 8, 9, 10$, and 12. At $2S = 14$ for $L = 6$ there are three allowed multiplets. The diagonalization of the Coulomb interaction gives the lowest state that avoids $\mathcal{R} = 1$ (CFP $\sim 10^{-7}$) and $\mathcal{R} = 3$ (CFP $< 10^{-2}$), and the next-lowest state that avoids $\mathcal{R} = 1$ (CFP $< 10^{-5}$) but orthogonality to the lowest state requires that it has significant parentage from $\mathcal{R} = 3$ (CFP ≈ 0.34).

One can see that the set of angular momentum multiplets L that can be constructed at a given value of $2l$ without parentage from pair states with $\mathcal{R} = 1$ is identical to the set of all allowed multiplets L at $2l^* = 2l - 4$. For a short-range repulsion (e.g. the Coulomb repulsion in the lowest Landau level), these multiplets will be (to a good approximation) the lowest energy eigenstates (the appropriate CFP for the actual eigenstates will be very small although not necessarily zero). More generally, in the lowest Landau level (remember that $l = S$), the set of multiplets L that can be constructed at given $2S$ without parentage from $\mathcal{R} \leqslant 2p - 1$ (i.e. with $\mathcal{R} \geqslant 2p + 1$ for all pairs; $p = 1, 2, \ldots$) is identical to the set of all allowed multiplets L at $2S^* = 2S - 2p(N - 1)$. The multiplets L forming the lowest Coulomb energy band at a given $2S$ are all multiplets allowed at $2S^*$. But $2S^* = 2S - 2p(N - 1)$ is just the effective magnetic monopole strength in the mean-field CF picture! Thus the CF picture with $2p$ attached flux quanta simply picks the subset of angular momentum multiplets which have no parentage from pair states with $\mathcal{R} \leqslant 2p - 1$, and neglects the long-range part of the pseudopotential, $V(\mathcal{R})$ for $\mathcal{R} \geqslant 2p + 1$.

6.4. Definition of the short-range pseudopotential

For systems containing more than three fermions in an angular momentum shell, the simple addition of angular momentum to determine the smallest possible L that has parentage from pair states with $L_{12} = 2l - 1$ is of no help. Instead, we make use of the following operator identity:

$$\hat{L}^2 + N(N - 2)\hat{l}^2 = \sum_{i<j} \hat{L}_{ij}^2. \tag{19}$$

Here $\hat{L} = \sum_i \hat{l}_i$ and $\hat{L}_{ij} = \hat{l}_i + \hat{l}_j$. The identity is easily proved by writing out the expression for $\hat{L}^2$ and for $\sum_{i<j} \hat{L}_{ij}^2$ and eliminating $\sum_{i<j} (\hat{l}_i \cdot \hat{l}_j)$ from the pair of equations. Taking matrix elements of equation (19) between states $|l^N, L\alpha\rangle$ described by equation (17) gives

$$L(L + 1) + N(N - 2)l(l + 1) = \langle l^N, L\alpha | \sum_{i<j} \hat{L}_{ij}^2 | l^N, L\alpha\rangle$$

$$= \frac{1}{2}N(N - 1)\sum_{L_{12}} \mathcal{G}_{L\alpha}(L_{12})L_{12}(L_{12} + 1) \tag{20}$$

where

$$\mathcal{G}_{L\alpha}(L_{12}) = \sum_{L'\alpha'} \left| G_{L\alpha, L'\alpha'}(L_{12}) \right|^2. \tag{21}$$

The coefficients of grandparentage satisfy the relation

$$\sum_{L_{12}} \sum_{L'\alpha'} G_{L\alpha, L'\alpha'}(L_{12}) G_{L\beta, L'\alpha'}(L_{12}) = \delta_{\alpha\beta}. \tag{22}$$

Of course, the energy of the multiplet $|l^N, L\alpha\rangle$ is given by

$$E_\alpha(L) = \frac{1}{2}N(N - 1)\sum_{L_{12}} \mathcal{G}_{L\alpha}(L_{12})V(L_{12}) \tag{23}$$

where $V(L_{12})$ is the electron pseudopotential.

It is important to make the following observations:

(i) The expectation value of $\sum_{i<j} \hat{L}_{ij}^2$ in a many-body state $|l^N, L\alpha\rangle$ increases as $L(L+1)$, but it is totally independent of α.

(ii) If the pseudopotential $V_H(L_{12})$ were a linear function of $\hat{L}_{12}^2$ (we refer to V_H as the 'harmonic pseudopotential'), all many-body multiplets with the same value of L would be degenerate.

(iii) The difference $\Delta V(L_{12}) = V(L_{12}) - V_H(L_{12})$ between the actual pseudopotential V and its harmonic part V_H lifts this degeneracy. If N_L many-body multiplets of V_H occur at angular momentum L, the anharmonic term ΔV in the pseudopotential causes them to 'repel one another' and results in a band of N_L non-degenerate multiplets.

Because the expectation value of $\sum_{i<j} \hat{L}_{ij}^2$ in a many-body state of angular momentum L increases as $L(L+1)$, a strict Hund's rule holds for harmonic pseudopotentials: for V_H that increases as a function of L_{12}, the highest energy state is always at the maximum possible value of L equal to $L^{\mathrm{MAX}} = Nl - N(N-1)/2$, and the lowest energy state is at the minimum allowed value of L equal to L^{MIN}. If V_H decreases as a function of L_{12}, the opposite occurs: the lowest energy state is at L^{MAX}, and the highest energy state is at L^{MIN} (this is a standard Hund's rule of atomic physics).

Neither of these Hund's rules may remain true in the presence of a large anharmonic term ΔV. For example, if the number of multiplets N_L at a value slightly larger than L^{MIN} is very large compared to $N_{L^{\mathrm{MIN}}}$, the strong level repulsion due to ΔV within this L-subspace can overcome the difference in the expectation values of V_H, and the lowest eigenvalue of V at L can be lower than that at L^{MIN}. However, only very few multiplets occur at large values of L: $N_{L^{\mathrm{MAX}}} = 1$ (for $M = L = L^{\mathrm{MAX}}$, the only state is $|l, l-1, \ldots, l-N+1\rangle$), $N_{L^{\mathrm{MAX}}-1} = 0$, $N_{L^{\mathrm{MAX}}-2} \leqslant 1$, $N_{L^{\mathrm{MAX}}-3} \leqslant 1$, etc. As a result, breaking of the Hund's rule that refers to the behaviour of energy at large L requires stronger anharmonicity than at small L. For the Coulomb pseudopotential in the lowest Landau level we always find that the highest energy does indeed occur at L^{MAX}. However, the ability to avoid parentage from pair states having large L_{ij} often favours many-body states at small $L > L^{\mathrm{MIN}}$ with large N_L, as prescribed by the CF picture.

The anharmonicity of the Coulomb pseudopotential in the lowest Landau level (which increases with increasing L_{12}) is critical for the behaviour of the FQH systems. We have found that the condition for the occurrence of subbands separated by gaps in the energy spectrum, and, in particular, for the occurrence of non-degenerate incompressible fluid ground states at specific values of the filling factor, is that the anharmonic term $\Delta V(L_{12})$ is positive and increases with increasing L_{12}. In other words, the total pseudopotential $V(L_{12})$ must increase more quickly than linearly as a function of $L_{12}(L_{12}+1)$.

6.5. Hidden symmetry of the short-range repulsion

From our numerical studies we have arrived at the following conjectures:

(i) The Hilbert space $\mathcal{H}_{Nl}$ of N identical fermions each with angular momentum l contains subspaces $\mathcal{H}_{Nl}^{(p)}$ of states that have no parentage from $\mathcal{R} \leqslant 2p - 1$. The subspaces $\tilde{\mathcal{H}}_{Nl}^{(p)} = \mathcal{H}_{Nl}^{(p)} \setminus \mathcal{H}_{Nl}^{(p+1)}$ can be defined; they hold states without parentage from $\mathcal{R} \leqslant 2p-1$, but with some parentage from $\mathcal{R} = 2p + 1$. Then

$$\mathcal{H}_{Nl} = \tilde{\mathcal{H}}_{Nl}^{(0)} \oplus \tilde{\mathcal{H}}_{Nl}^{(1)} \oplus \tilde{\mathcal{H}}_{Nl}^{(2)} \oplus \cdots. \tag{24}$$

(ii) For an 'ideal' short-range repulsive pseudopotential V_{SR}, for which $V_{SR}(\mathcal{R}) \gg V_{SR}(\mathcal{R}+2)$, the huge difference between energy scales associated with different pair states results in the following (dynamical) symmetry:

   (a) subspaces $\tilde{\mathcal{H}}_{Nl}^{(p)}$ are the interaction eigensubspaces,
   (b) p is a good quantum number,
   (c) the energy spectrum splits into bands (larger p corresponds to lower energy), and
   (d) the energy gap above the pth band scales as $V(2p-2) - V(2p)$.

(iii) For a finite short-range pseudopotential V (increasing more quickly than V_H as a function of L_{12}), the above symmetry is only approximate, but the correlation between energy and parentage from highly repulsive pair states persists, and so do the gaps in the energy spectrum. The mixing between neighbouring subbands is weak, although the structure of energy levels within each subband depends on the form of $V(L_{12})$ at $\mathcal{R} \geqslant 2p+1$.

(iv) The set of angular momentum multiplets in subspace $\mathcal{H}_{Nl}^{(p)}$ is identical to $\mathcal{H}_{Nl^*}$, where $l^* = l - p(N-1)$.

Although at present we do not have a general analytic proof for the last conjecture, we have verified it for various small systems and have not found one for which it would fail.

   The above conjectures can be immediately translated into the planar geometry. The harmonic pseudopotential $V_H(m)$, used to define the class of short-range pseudopotentials, is that of a repulsive interaction potential $V(r)$ which is linear in r^2. Then,

$$\mathcal{H}_\nu = \tilde{\mathcal{H}}_\nu^{(0)} \oplus \tilde{\mathcal{H}}_\nu^{(1)} \oplus \tilde{\mathcal{H}}_\nu^{(2)} \oplus \cdots \tag{25}$$

where $\mathcal{H}_\nu$ is the Hilbert space of electrons filling a fraction ν of an infinitely degenerate Landau level, and subspaces $\tilde{\mathcal{H}}_\nu^{(p)}$ contain states without parentage from $m \leqslant 2p-1$, but with some parentage from $m = 2p+1$. The (approximate) dynamical symmetry holds for the Coulomb interaction, and the low-energy band $\mathcal{H}_\nu^{(p)}$ contains the same angular momentum multiplets as $\mathcal{H}_{\nu^*}$, with ν^* defined by the CF prescription in equation (16).

   The validity of our conjectures for systems interacting through the Coulomb pseudopotential is illustrated in figure 4 for four electrons in the lowest Landau level at $2S = 5, 11, 17,$ and 23. Different symbols mark bands corresponding to (approximate) subspaces $\mathcal{H}_{Nl}^{(p)}$ with different p. The same sets of multiplets reoccur for different $2S$ in bands related by $\mathcal{H}_{Nl}^{(p)} \sim \mathcal{H}_{Nl^*}$.

6.6. Comparison with atomic shells: Hund's rule

Our conjectures (verified by the numerical experiments) are based on the behaviour of systems of interacting fermions partially filling a shell of degenerate single-particle states of angular momentum l. This is a central problem in atomic physics and in nuclear shell-model studies of energy spectra. It is interesting to compare the behaviour of the spherical harmonics of atomic physics with that of the monopole harmonics considered here. For monopole harmonics $l = S+n$, where S is half of the monopole strength (and can be integral or half-integral) and n is a non-negative integer. For the lowest angular momentum shell, $l = S$. For spherical harmonics, $S = 0$ and $l = n$. If in each case electrons are confined to a 2D spherical surface of radius R, one can evaluate the pair interaction energy V as a function of the pair angular momentum L_{12}. The resulting pseudopotentials, $V(\mathcal{R})$ for the FQH system in the lowest Landau level, and $V(L_{12})$ for atomic shells in a zero magnetic field, are shown in figure 5 for a few small values of l. In obtaining these results we have restricted ourselves to spin-polarized shells, so only orbital angular momentum is considered. It is clear that in the case of spherical harmonics the largest pseudopotential coefficient occurs for the lowest pair angular momentum, exactly
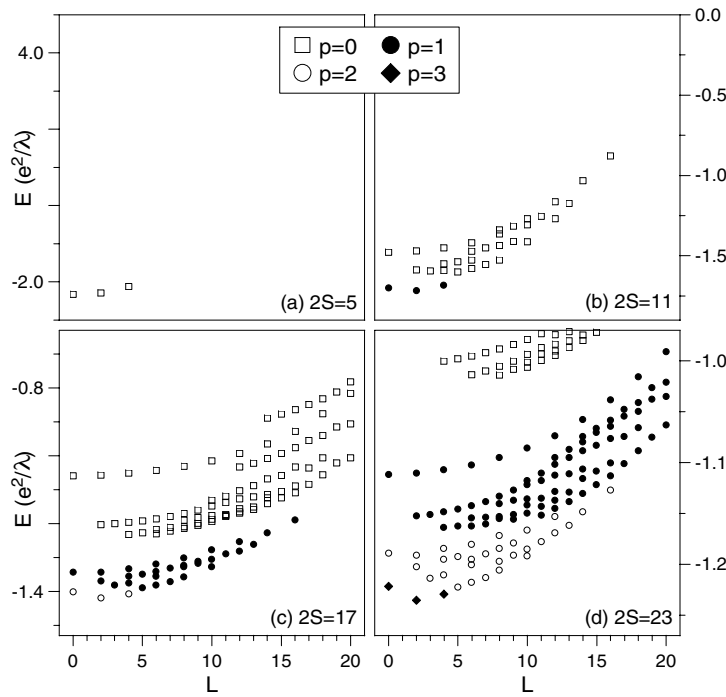
Figure 4. The energy spectra of four electrons in the lowest Landau level calculated on a Haldane sphere with $2S = 5$, 11, 17, and 23. All those values of $2S$ are equivalent in the mean-field composite-fermion picture (the Chern–Simons transformation with $p = 0, 1, 2,$ and 3, respectively). Different symbols mark states with different numbers of avoided pair states with highest energy.
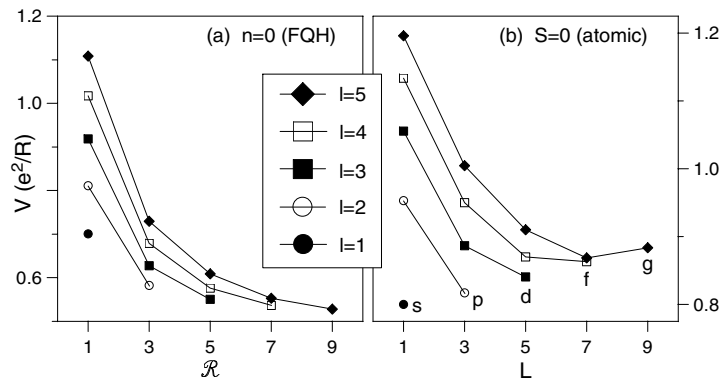

Figure 5. The pseudopotentials V of the Coulomb interaction for the pair of electrons each with angular momentum l: (a) lowest Landau level on a Haldane sphere, monopole harmonics, $n = 0$, $l = S$, V plotted as a function of relative pair angular momentum $\mathcal{R}$; (b) atomic shell, spherical harmonics, $S = 0$, $l = n$, V plotted as a function of pair angular momentum L.

the opposite of what occurs for monopole harmonics. As a consequence of equation (19), which relates the total angular momentum L to the average pair angular momentum L_{12}, the standard atomic Hund's rule predicts that the energy of a few-electron system in an atomic shell will, on the average, decrease as a function of total angular momentum, which is opposite to the behaviour of energy of electrons in the lowest Landau level. The difference between the

energy spectra of electrons interacting through atomic and FQH pseudopotentials of figure 5 is demonstrated in figure 6, where we plot the result for four electrons in shells of angular momentum $l = 3$ and 5. The solid circles correspond to monopole harmonics and the open ones to spherical harmonics. Note that at L^{MAX} the former give the highest energy and the latter the lowest. Due to anharmonicity of the pseudopotentials, the behaviour of energy at low L does not always follow a simple Hund's rule for either FQH or atomic system. The FQH ground state for $l = 3$ occurs at $L = 0$ (this is the $\nu = 2/3$ incompressible state). However, for $l = 5$, the lowest of the three states at $L = 2$ has lower energy than the only state at $L = 0$. This ground state at $L = 2$ contains one quasihole in the Laughlin $\nu = 1/3$ state and it is the only four-electron state at this filling in which electrons can avoid parentage from the $\mathcal{R} = 1$ pair state. Exactly the opposite happens for the atomic system at $l = 5$, where the anharmonicity is able to push the highest of the three $L = 2$ states above the high-energy state at $L = 0$.
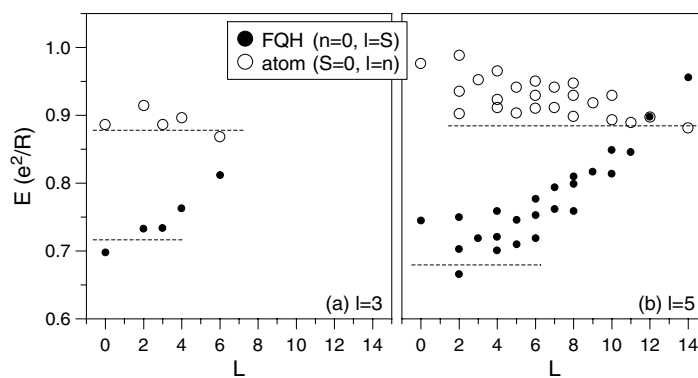
Figure 6. The energy spectra of four electrons in a degenerate shell of angular momentum $l = 3$ (a) and $l = 5$ (b), interacting through the pseudopotentials of figure 5: open circles—atomic shell ($S = 0$ and $l = n$); solid circles—lowest Landau level ($n = 0$ and $l = S$).

6.7. Higher Landau levels

Thus far we have considered only the lowest angular momentum shell (lowest Landau level) with $l = S$. The interaction of a pair of electrons in the nth excited shell of angular momentum $l = S + n$ can easily be evaluated to obtain the pseudopotentials $V(L_{12})$ shown in figure 7. Here we compare $V_n(L_{12})$ as a function of $L_{12}(L_{12}+1)$ for $n = 0, 1$, and 2. It can readily be observed that $V_{n=0}$ increases more quickly than $L_{12}(L_{12} + 1)$ over the entire range of L_{12}, while $V_{n=1}$ and $V_{n=2}$ do so only up to a certain value of L_{12} (i.e., above a certain value of $\mathcal{R} = 2l - L_{12}$). For $n = 1$, $V_{n=1}$ has short range for $\mathcal{R} \geqslant 3$ but is essentially linear in $L_{12}(L_{12}+1)$ from $\mathcal{R} = 1$ to 5. For $n = 2$, $V_{n=2}$ has short range for $\mathcal{R} \geqslant 5$ but is sublinear in $L_{12}(L_{12} + 1)$ from $\mathcal{R} = 1$ to 7. More generally, we find that the pseudopotential in the nth excited shell (Landau level) has short range for $\mathcal{R} \geqslant 2n + 1$.

Because the conclusions of the CF picture depend so critically on the short range of the pseudopotential, they are not expected to be valid for all fractional fillings of higher Landau levels. For example, the ground state at $\nu = 2 + 1/3 = 7/3$ does not have Laughlin-type correlations (i.e. electrons in the $n = 1$ Landau level do not avoid parentage from $\mathcal{R} = 1$) even if it is non-degenerate ($L = 0$) and incompressible (as found experimentally [24]).
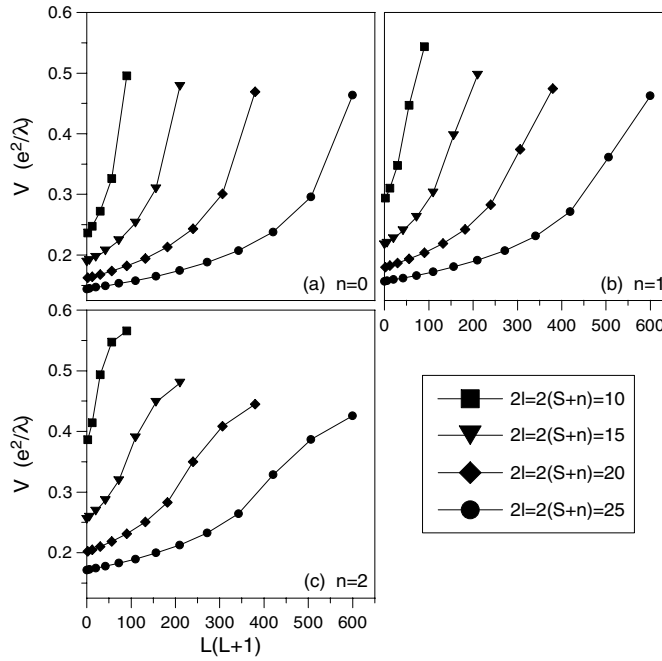
Figure 7. The pseudopotentials V of the Coulomb interaction in the lowest (a), first excited (b), and second excited (c) Landau levels, calculated on a Haldane sphere, as functions of the squared pair angular momentum $L(L+1)$. Different symbols correspond to different Landau-level degeneracies $2l + 1$.

7. Fermi-liquid model of composite fermions

The numerical results of the type shown in figure 1 have been understood in a very simple way using Jain's composite-fermion picture. For the ten-particle system, the Laughlin $\nu = 1/3$ incompressible ground state at $L = 0$ occurs for $2S = 3(N - 1) = 27$. The low-lying excited states consist of a single QP pair, with the QE and QH having angular momenta $l_{QE} = 11/2$ and $l_{QH} = 9/2$. In the mean-field CF picture, these states should form a degenerate band of states with angular momentum $L = 1, 2, \ldots, 10$. More generally, $l_{QE} = (N + 1)/2$ and $l_{QH} = (N - 1)/2$ for the Laughlin state of an N-electron system, and the maximum value of L is N. The energy of this band would be $E = \hbar\omega_c^* = \hbar\omega_c/3$, the effective CF cyclotron energy needed to excite one CF from the (completely filled) lowest to the (completely empty) first excited CF Landau level. From the numerical results, two shortcomings of the mean-field CF picture are apparent. First, due to the QE–QH interaction (neglected in the CF picture), the energy of the QE–QH band depends on L, and the 'magnetoroton' QE–QH dispersion has a minimum at $L = 5$. Second, the state at $L = 1$ either does not appear, or is part of the continuum (in an infinite system) of higher energy states above the magnetoroton band.

At $2S = 27 - 1 = 26$ and $2S = 27 + 1 = 28$, the ground state contains a single quasiparticle (QE or QH, respectively), whose angular momenta $l_{QE} = l_{QH} = N/2 = 5$ result from the CS transformation which gives $2S^* = 2S - 2(N - 1) = 8$ for QE and 10 for QH (and $l_{QE} = S^* + 1$ and $l_{QH} = S^*$). States containing two identical QPs form lowest energy bands at $2S = 25$ (two QEs) and $2S = 29$ (two QHs). The allowed angular momenta of two identical CF QPs (which are fermions) each with angular momentum l_{QP} are $L = 2l_{QP} - j$ where j is an odd integer. Of course, l_{QP} depends on $2S$ in the CF picture, and at $2S = 25$

we have $l_{QE} = S^* + 1 = S - (N - 1) + 1 = 9/2$ yielding $L = 0, 2, 4, 6$, and 8, while at $2S = 29$ we have $l_{QH} = S^* = S - (N - 1) = 11/2$ and $L = 0, 2, 4, 6, 8, 10$. More generally, $l_{QE} = (N - 1)/2$ and $l_{QH} = (N + 1)/2$ in the 2QE and 2QH states of an N-electron system, and the maximum values of L are $N - 2$ for QEs and N for QHs. As for the magnetoroton band at $2S = 27$, the CF picture does not account for QP interactions and incorrectly predicts the degeneracy of the bands of 2QP states at $2S = 25$ and 27.

The energy spectra of states containing more than one CF quasiparticle can be described in the following phenomenological Fermi-liquid picture. The creation of an elementary excitation, QE or QH, in a Laughlin incompressible ground state requires a finite energy, ε_{QE} or ε_{QH}, respectively. In a state containing more than one Laughlin quasiparticle, QEs and/or QHs interact with one another through appropriate QE–QE, QH–QH, and QE–QH pseudopotentials.

An estimate of the QP energies can be obtained by comparing the energy of a single QE (for the ten-electron system, the energy of the ground state at $L = N/2 = 5$ for $2S = 27 - 1 = 26$) or a single QH (the $L = N/2 = 5$ ground state at $2S = 27 + 1 = 28$) with the Laughlin $L = 0$ ground state at $2S = 27$. There can be finite-size effects here, because the QP states occur at different values of $2S$ to the ground state, but using the correct magnetic length $\lambda = R/\sqrt{S}$ (R is the radius of the sphere) in the unit of energy e^2/λ at each value of $2S$, and extrapolating the results as a function of N^{-1} to an infinite system, should give reliable estimates of ε_{QE} and ε_{QH} for a macroscopic system.

The QP pseudopotentials V_{QP-QP} can be obtained by subtracting from the energies of the 2QP states obtained numerically at $2S = 25$ (2QE), $2S = 27$ (QE–QH), and $2S = 29$ (2QH) the energy of the Laughlin ground state at $2S = 27$ and two energies of appropriate non-interacting QPs. As for the single QP, the energies calculated at different $2S$ must be taken in correct units of $e^2/\lambda = \sqrt{S}e^2/R$ to avoid finite-size effects. This procedure was carried out in references [18, 26].

In figure 8 we plot the QE–QE and QH–QH pseudopotentials for Laughlin $\nu = 1/3$ and 1/5 states. As we have seen for two electrons (see figure 3), the angular momentum L_{12} of a pair of identical fermions in an angular momentum shell (or a Landau level) is quantized, and the convenient quantum number to label the pair states is $\mathcal{R} = 2l_{QP} - L_{12}$ (on a sphere) or relative (REL) angular momentum m (on a plane). When plotted as a function of $\mathcal{R}$, the pseudopotentials calculated for systems containing between six to eleven electrons (and thus for different QP angular momenta l_{QP}) behave similarly and, for $N \to \infty$ (i.e., $2S \to \infty$), they seem to converge to the limiting pseudopotentials $V_{QP-QP}(\mathcal{R} = m)$ describing an infinite planar system.

In figure 9 we plot the QE–QH pseudopotentials for Laughlin $\nu = 1/3$ and 1/5 states. As for a conduction electron and a valence hole pair in a semiconductor (an exciton), the motion of a QE–QH pair which does not carry a net electric charge is not quantized in a magnetic field. The appropriate quantum number to label the states is the continuous wavevector k (or momentum), which on a sphere is given by $k = L/R = L/\lambda\sqrt{S}$ (remember that L is given in units of $\hbar$). When plotted as a function of k, the pseudopotentials calculated for systems containing between six and eleven electrons fall on the same curve that describes a continuous magnetoroton dispersion $V_{QE-QH}(k)$ of an infinite planar system (the lines in figure 9 are only to guide the eye). Only the energies for $L \geqslant 2$ are shown in figure 9, since the single QE–QH pair state at $L = 1$ is either disallowed (hard core) or falls into the continuum of states above the magnetoroton band. The magnetoroton minima for the Laughlin $\nu = 1/3$ and 1/5 states occur at about $k_0 = 1.4 \lambda^{-1}$ and $k_0 = 1.1 \lambda^{-1}$, respectively. The magnetoroton band at $\nu = 1/3$ is well decoupled from the continuum of higher states because the bandwidth $\sim 0.05 e^2/\lambda$ is much smaller than the energy gap $\varepsilon_{QE} + \varepsilon_{QH} = 0.1 e^2/\lambda$ for additional QE–QH pair excitations. At
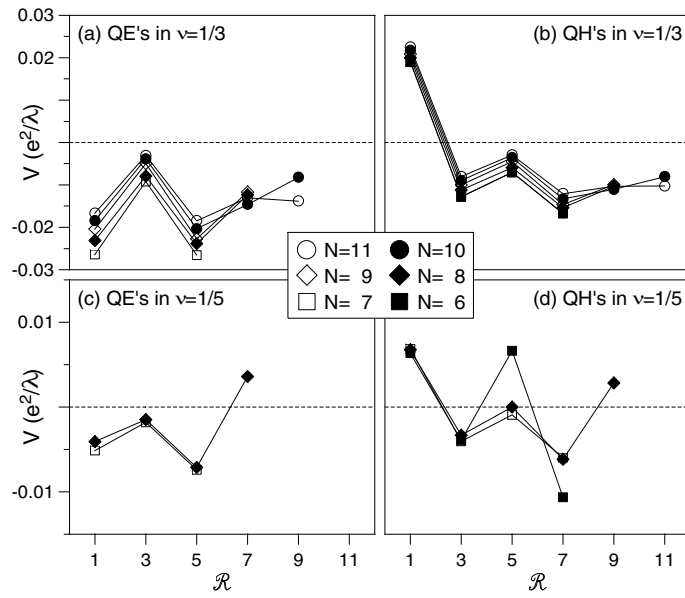
Figure 8. The pseudopotentials of a pair of quasielectrons (left) and quasiholes (right) in Laughlin $\nu = 1/3$ (top) and $\nu = 1/5$ (bottom) states, as functions of relative pair angular momentum $\mathcal{R}$. Different symbols mark data obtained in the diagonalization of between six and eleven electrons.
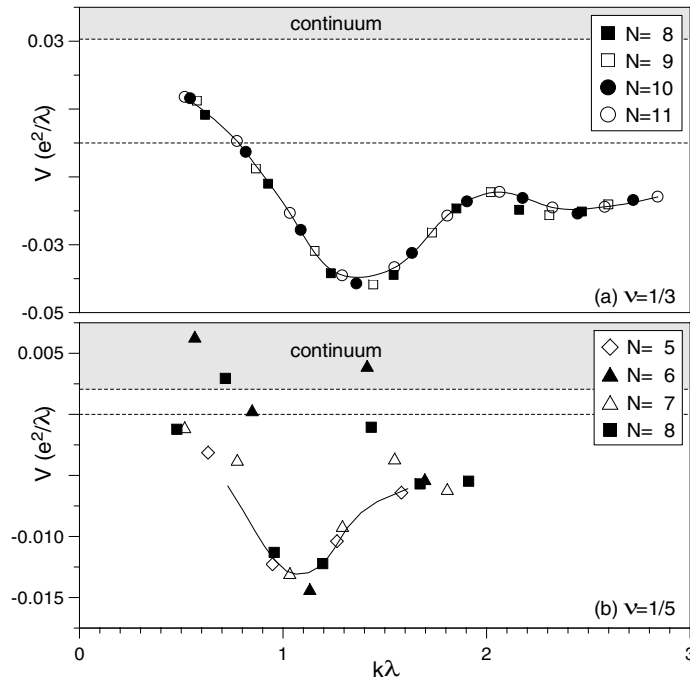

Figure 9. The pseudopotentials of a quasielectron–quasihole pair in Laughlin $\nu = 1/3$ (a) and $\nu = 1/5$ (b) states as functions of wavevector k. Different symbols mark data obtained in the diagonalization of between five and eleven electrons.

$\nu = 1/5$, the bandwidth $\sim 0.015e^2/\lambda$ is closer to the single-particle gap $\varepsilon_{QE} + \varepsilon_{QH} = 0.021e^2/\lambda$ and the state of two magnetorotons each with $k \approx k_0$ can couple to the highest-enrgy QE–QH pair states at $k \leqslant 2k_0$.

Knowing the QP–QP pseudopotentials and the bare QP energies allows us to evaluate the energies of states containing three or more QPs. Typical results are shown in figure 10. In frame (a) we show the energy spectra of three QEs in the Laughlin $\nu = 1/3$ state of eleven electrons. The spectrum in frame (b) gives energies of three QHs in the nine-electron system at the same filling. The exact numerical results obtained in diagonalization of the eleven- and nine-electron systems are represented by plus signs and the Fermi-liquid-picture results are marked by solid circles. The exact energies above the dashed lines correspond to higher energy states that contain additional QE–QH pairs. It should be noted that in the mean-field CF picture which neglects the QP–QP interactions, all of the 3QP states would be degenerate and the energy gap separating the 3QP states from higher states would be equal to $\hbar\omega_c^* = \hbar\omega_c/3$. Although the fit in figure 10 is not perfect, it is quite good and justifies the use of the Fermi-liquid picture to describe (compressible) states at $\nu \neq (2p + 1)^{-1}$.
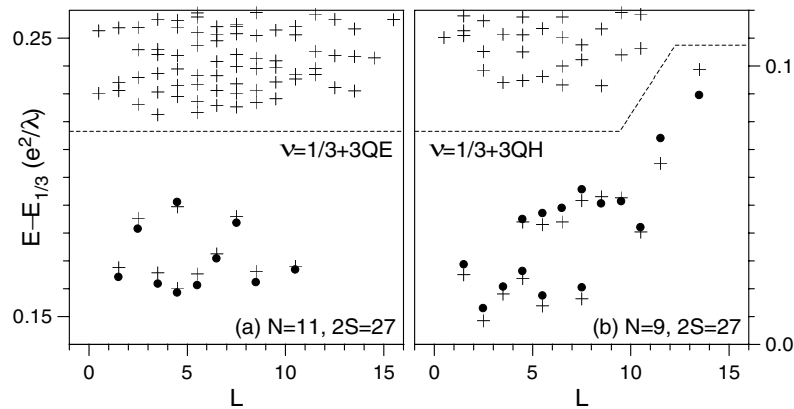

Figure 10. The energy spectra of three quasielectrons (a) and three quasiholes in the Laughlin $\nu = 1/3$ state. The crosses correspond to the Fermi-liquid calculation using pseudopotentials from figures 8(a), 8(b); the solid circles give exact spectra obtained in full diagonalization of the Coulomb interaction of eleven (a) and nine (b) electrons.

8. Composite-fermion hierarchy

The sequence of Laughlin–Jain states with filling factor ν given by $\nu = \nu^*(1 + 2p\nu^*)^{-1}$ where $p = 1, 2, \ldots$, and the CF filling factor ν^* is any non-zero integer, is the most prominent set of condensed states observed experimentally. However, this sequence (together with the conjugate 'hole' states, $\nu \to 1 - \nu$) does not contain all odd-denominator fractions the way the Haldane hierarchy scheme does. The question arises quite naturally of how to treat the CF values of ν^* which are not integers. The answer leads to the CF approach to the hierarchy of incompressible quantum fluid ground states [15].

Consider a state of N_0 electrons at a monopole strength $2S_0$ with a filling factor ν_0. The CS transformation that attaches to each electron $2p_0$ flux quanta oriented opposite to the applied magnetic field results in the CF system at an effective filling factor ν_0^* given by $(\nu_0^*)^{-1} = \nu_0^{-1} - 2p_0$ and an effective monopole strength $2S_0^* = 2S_0 - 2p_0(N_0 - 1)$. The procedure for handling non-integral values of CF filling factor ν_0^* is to set it equal to

$\nu_0^* = n_1 + \nu_1$, where n_1 is an integer and ν_1 is the fractional filling of the CF quasiparticle level (same sign as n_1 for QEs and opposite for QHs). Our problem is then that of placing N_1 quasiparticles into $2l_1 + 1$ available states of a CF shell (Landau level) of angular momentum l_1: the QEs into the lowest empty shell with $l_1 = |S_0^*| + n_1 + 1$, or the QHs into the highest filled shell with $l_1 = |S_0^*| + n_1$, We now ignore all completely filled and completely empty CF shells, and reapply the CS transformation by setting $S_1 = l_1$ and attaching $2p_1$ flux quanta to each of the N_1 quasiparticles in the partially filled CF shell. This produces a new type of QPs and a new QP filling factor ν_1^* given by $(\nu_1^*)^{-1} = \nu_1^{-1} - 2p_1$. If ν_1^* is an integer, we obtain daughter states in the hierarchy. If it is not, we write $\nu_1^* = n_2 + \nu_2$, where ν_2 represents the partial filling of the new QP shell, and repeat the mean-field CF procedure. This leads to the set of equations

$$\nu_l^{-1} = 2p_l + (n_{l+1} + \nu_{l+1})^{-1} \qquad (26)$$

where ν_l is the QP filling factor and $2p_l$ is the number of flux quanta attached to each fermion at the lth level of the CF hierarchy.

As an example, consider a system of $N_0 = 12$ electrons at $2S_0 = 30$. We apply the mean-field CF approximation by attaching to each electron $2p_0 = 2$ flux quanta. This gives the effective CF monopole strength $2S_0^* = 30 - 2(12 - 1) = 8$. The lowest CF shell is filled with nine particles, and there are $N_1 = 3$ quasielectrons in the first excited ($n_1 = 1$) CF shell of angular momentum $l_1 = 5$. The filling factor at this level of hierarchy is $\nu_0^* = 1 + \nu_1$. We now reapply the CF transformation by attaching $2p_1 = 4$ flux quanta to each of $N_1 = 3$ QEs at $2S_1 = 10$ and obtain $2S_1^* = 10 - 4(3 - 1) = 2$. The lowest CF shell of $l_1 = 1$ is now completely filled yielding $\nu_1^* = 1$. Using equation (26) we obtain $\nu_1^{-1} = 4 + 1^{-1} = 5$ and $\nu_0^{-1} = 2 + (1 + 1/5)^{-1} = 17/6$.

If the mean-field CF picture worked on all levels of hierarchy, the twelve-electron system at $2S = 30$ should have an incompressible $L = 0$ ground state corresponding to the filling factor $\nu = 6/17$. In figure 11(a) we show the low-energy sector of the spectrum calculated for this system using the Fermi-liquid picture (only the lowest energy states containing three QEs in the Laughlin $\nu = 1/3$ state are calculated). Indeed, the $\nu = 6/17$ hierarchy ground state at $L = 0$ is separated from higher states by a small gap in the twelve-electron spectrum
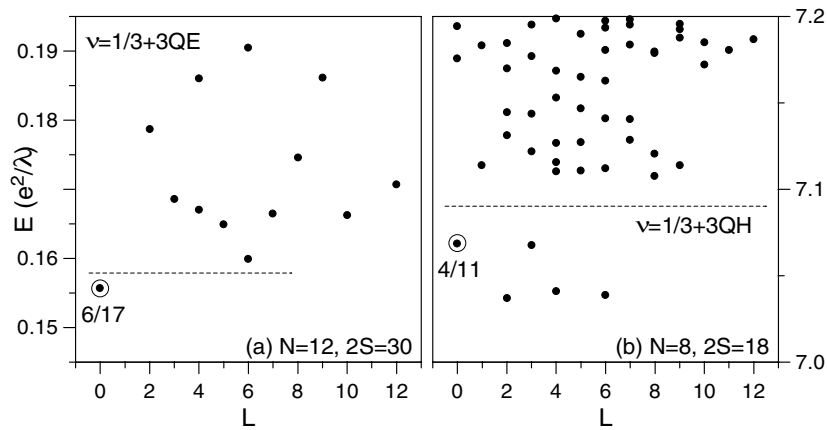

Figure 11. (a) The low-energy spectrum of three quasielectrons in the Laughlin $\nu = 1/3$ state of twelve electrons calculated using the quasielectron pseudopotential from figure 8(a). (b) The energy spectrum of three quasiholes in the Laughlin $\nu = 1/3$ state of eight electrons obtained in full diagonalization of the electron–electron Coulomb interaction.

(although it is not clear that this small gap will persist in the thermodynamic limit [26]).

Though the CF hierarchy picture seems to work in some cases, there are others where it is clearly in complete disagreement with numerical results. For example, a CF transformation with $2p_0 = 2$ applied to an $(N_0 = 8)$-electron system at $2S_0 = 18$ gives $2S_0^* = 18 - 2(8-1) = 4$, $n_1 = 1$, and $N_1 = 3$ QEs left in the shell with $l_1 = 3$. Adding the three QE angular momenta of $l_1 = 3$ gives a low-energy band at $L = 0, 2, 3, 4$, and 6. Reapplication of the CF transformation with $2p_1 = 2$ gives $2S_1^* = 6 - 2(3 - 1) = 2$, i.e. the completely filled lowest shell, $v_1^* = 1$ ($n_2 = 1$ and $v_2 = 0$). From equation (26) we get $v_1 = 1/3$ and $v_0 = 4/11$. In figure 11(b) we show the spectrum obtained by exact numerical diagonalization of an eight-electron system at $2S = 18$. It is apparent that the set of multiplets at $L = 0, 2, 3, 4$, and 6 form the low-energy band. However, the reapplication of the mean-field CF transformation to the three QEs in the $l_1 = 3$ shell (which predicts an $L = 0$ incompressible ground state corresponding to $v = 4/11$) is definitely wrong.

The reason why the CF hierarchy picture does not always work is not difficult to understand. The electron (Coulomb) pseudopotential in the lowest Landau level $V_e(\mathcal{R})$ satisfies the 'short-range' criterion (i.e. it increases more quickly with decreasing $\mathcal{R}$ than the harmonic pseudopotential V_H) in the entire range of $\mathcal{R}$, which is the reason for the incompressibility of the principal Laughlin $v = (2p + 1)^{-1}$ states. However, this does not generally hold for the QP pseudopotentials on higher levels of the hierarchy. In figure 8 we plotted $V_{QE-QE}(\mathcal{R})$ and $V_{QH-QH}(\mathcal{R})$ for the $v = 1/3$ and $v = 1/5$ Laughlin states of six to eleven electrons. Clearly, the QE and QH pseudopotentials are quite different and neither one decreases monotonically with increasing $\mathcal{R}$. On the other hand, the corresponding pseudopotentials in $v = 1/3$ and $1/5$ states look similar, only the energy scale is different. The convergence of energies at small $\mathcal{R}$ obtained for larger N suggests that the maxima at $\mathcal{R} = 3$ for QEs and at $\mathcal{R} = 1$ and 5 for QHs, as well as the minima at $\mathcal{R} = 1$ and 5 for QEs and at $\mathcal{R} = 3$ and 7 for QHs, persist in the limit of large N (i.e. for an infinite system on a plane). Consequently, the only incompressible daughter states of Laughlin $v = 1/3$ and $1/5$ states are those with $v_{QE} = 1$ or $v_{QH} = 1/3$ and (maybe) $v_{QE} = 1/5$ and $v_{QH} = 1/7$. It is clear that no incompressible daughter states of the parent Laughlin $v = 1/3$ state will form at e.g. $v = 4/11$ ($v_{QE} = 1/3$) or $4/13$ ($v_{QH} = 1/5$), but that they will form (at least, in finite systems [26]) at $v = 6/17$ ($v_{QE} = 1/5$) or $6/19$ ($v_{QH} = 1/7$).

From the CF hierarchy scheme we find the Jain–Laughlin states when the CS transformation is applied directly to electrons (or to holes in a more-than-half-filled level). These states occur at integral values of v^*, the effective CF filling factor, and correspond to completely filling a QP shell. For example, the $v = 2/5$ state occurs when $v^* = 2$, and the CFs in the first excited shell (which are Laughlin QEs of the $v = 1/3$ state) have $v_{QP} = 1$. The angular momenta of the two lowest CF shells are $l_0^* = |S^*|$ and $l_1^* = |S^*| + 1$, so they contain $2l_0^* + 1$ and $2l_1^* + 1$ states, respectively. Since $v_{QP} = 1$, there are $N_{QP} = 2l_1^* + 1$ CF quasiparticles. The total number of states filled by the N fermions is $(2l_0^* + 1) + (2l_1^* + 1) = 2N_{QP} - 2$, giving $N = 2N_{QP} - 2$. For an infinite system this is just Haldane's relation between the number of quasiparticles and the number of electrons, $N = 2qN_{QP}$, for the integer $q = 1$. This demonstrates that integrally filled CF shells correspond to $v_{QP} = 1$, a completely filled shell of Laughlin QPs. Adding new fermions to a system with $v_{QP} = 1$ requires creating a new type of QPs, and the counting of available QP states turns out to be exactly the same in the CF hierarchy and Haldane's boson hierarchy pictures. Integral CF filling (i.e., $v_{QP} = 1$) gives a valid mean-field picture independent of QP–QP interactions provided that the gap for creating new QPs is positive. When v^* is non-integral, the mean-field picture is valid only at values of L for which the 'short-range' requirement on the pseudopotential $V_{QP-QP}(L)$ is satisfied. The form of the QP–QP interactions obtained from our numerical calculations makes it clear that

the mean-field approximation is not valid at certain quasiparticle fillings (e.g. for $\nu_{QP} = 1/3$ filling of the quasielectron levels of the electron $\nu = 1/3$ state).

9. Systems containing electrons and valence band holes

There has been a great deal of interest in photoluminescence (PL) of 2D systems in high magnetic fields. An important ingredient in understanding PL is the negatively charged exciton (X^-). The X^- consists of two electrons bound to a valence band hole. If the total spin of the pair of electrons, J_e, is zero, the X^- is said to be a singlet (X_s^-); if $J_e = 1$ the X^- is called a triplet (X_t^-). Only the X_s^- is bound in the absence of a magnetic field, but in infinite magnetic field (where only a single Landau level is relevant) only the X_t^- is bound in a 2D system. It often occurs that the photoexcited hole is separated from the plane of the electron system by a small distance (this can happen, e.g., in wide GaAs quantum wells when the electron gas is confined to one GaAs/AlGaAs interface by remote ionized donors, and the photoexcited holes reside close to the other GaAs/AlGaAs interface). Several remarkable effects associated with electron–hole systems and charged excitons can be understood using the composite-fermion picture.

9.1. Charged excitons and the hidden symmetry in the lowest Landau level

First let us consider the idealized 2D system at so large a magnetic field that only the lowest electron and hole Landau levels need be considered. The energy spectrum for a two-electron–one-hole system at $2S = 20$ is shown in figure 12. The triplet X^- with angular momentum $l_{X^-} = S - 1$ is the only bound state, with binding energy $\sim 0.05 e^2/\lambda$. A pair of (unbound) singlet and triplet states occur at the energy equal exactly to the exciton energy E_X. In these so-called 'multiplicative' states a neutral exciton X in its ground state is decoupled from the second electron. Addition of exciton and electron angular momenta $L_X = 0$ and $l_e = S$ gives a state of total angular momentum $L = S$, and addition of two electron spins of $1/2$ gives both $J_e = 0$ and 1 spin configurations.

   The occurrence of unbound states at $E = E_X$ and $L = S$ is a manifestation of the following 'hidden symmetry': because of the exact overlap of electron and hole orbitals in the lowest Landau level (scaled with the same magnetic length λ), and thus independence of the strength of interaction of the type of particles involved, the commutator of an operator $d_X^\dagger$ that creates an exciton in its $L_X = 0$ ground state (on a sphere, $d_X^\dagger = \sum_m (-1)^m c_m^\dagger h_m^\dagger$, where $c_m^\dagger$ and $h_m^\dagger$ are electron and hole creation operators), with the interaction Hamiltonian H, is $[H, d_X^\dagger] = E_X d_X^\dagger$. As a result, if Ψ is an eigenstate of N_e electrons and N_h holes with an eigenenergy E and angular momentum quantum numbers L and M, then the multiplicative state $d_X^\dagger \Psi$ of $N_e + 1$ electrons and $N_h + 1$ holes is also an eigenstate with energy $E + E_X$ and the same L and M. A good quantum number conserved due to the 'hidden symmetry' is the number of decoupled excitons, N_X. In particular, the ground state for $N_e = N_h = N$ is the totally multiplicative state $(d_X^\dagger)^N |vac\rangle$ with $N_X = N$; for an infinite system this ground state can be viewed as a Bose condensate of non-interacting excitons. It can be readily found that the application of the PL operator that annihilates an optically active exciton (d_X) reduces its N_X by one, and therefore that only the multiplicative electron–hole states with $N_X > 0$ are optically active (have non-vanishing PL intensity). In figure 12, the two multiplicative states at $E = E_X$ and $L = S$ have $N_X = 1$, and all others have $N_X = 0$.

   It is essential to realize that two independent symmetries forbid the recombination of a triplet X^- ground state in figure 12:
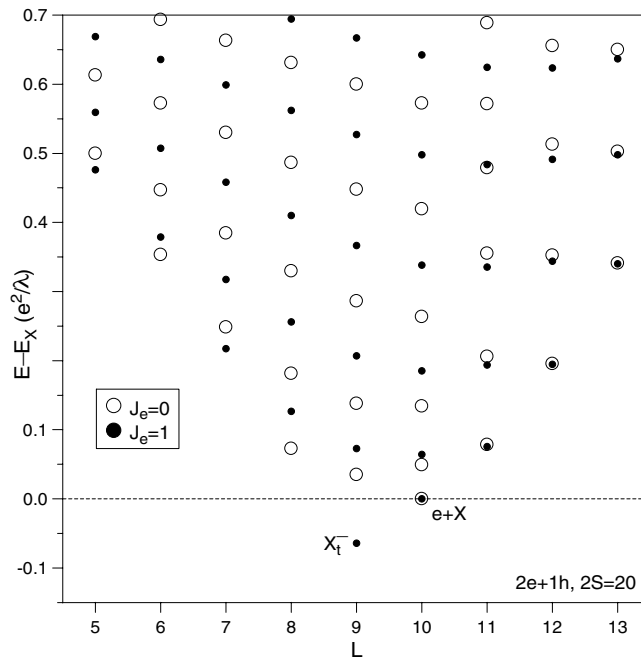
Figure 12. The energy spectrum (binding energy versus angular momentum) of a two-electron–one-hole system in the lowest Landau level at $2S = 20$. Open and solid circles mark singlet and triplet spin configurations, respectively.

- Due to the 2D translational/rotational space invariance, the PL operator d_X conserves two angular momentum quantum numbers. On a sphere, these are L and M, and the resulting optical selection rule allows only a state with $L = S$ to decay by e–h recombination. On a plane, these are the projection of total angular momentum (M) and an additional angular momentum quantum number K associated with partial decoupling of the centre-of-mass motion of a charged system in a homogeneous magnetic field, and the radiative channel for an X^- is that of $M + K = 0$. This (geometrical) symmetry can be broken by collisions, but persists in systems with a finite quantum well width, finite electron and hole layer separation, or Landau-level mixing.
- Due to the equal strength of e–e, h–h, and e–h interactions, N_X is a good quantum number. Since N_X is decreased in a PL process, only the multiplicative ($N_X > 0$) states are radiative. This (dynamical) symmetry is not broken by collisions, and requires breaking electron–hole orbital symmetry.

Since a number of independent factors are needed to allow for the recombination of a triplet X^-, this complex (in narrow and symmetric quantum wells and in high magnetic fields) is expected to be a well-defined long-lived quasiparticle. The correlations, optical properties, etc are expressed more easily in terms of this quasiparticle than in terms of individual electrons and holes. The finite angular momentum of an X^- in spherical geometry (partial decoupling of the centre-of-mass excitations from the relative motion on a plane) can be viewed as the formation of a degenerate Landau level of this (charged) quasiparticle. As will be shown later, the interaction of X^- quasiparticles with one another and with electrons can be described using the ideas familiar in the context of FQH systems (Laughlin correlations, composite fermions, parentage, etc).

9.2. Interaction of charged excitons

The simplest system in which to study X^-–X^- interaction contains four electrons and two holes. Its energy spectrum at $2S = 17$ is shown in figure 13. The low-energy spectrum is characterized by four bands which we have identified as follows:

  (i) The lowest band taking on all even values between $L = 0$ and 12 consists of a pair of charged excitons X^- (each with angular momentum $l_{X^-} = S - 1$).
 (ii) The next band contains an electron with $l_e = S$ and a negatively charged biexciton X_2^- (a bound state of an X and an X^-) with angular momentum $l_{X_2^-} = S - 2$; the allowed L-values go from $|l_e - l_{X_2^-}| = 2$ to $l_e + l_{X_2^-} - 1 = 14$.
(iii) A band of multiplicative states containing an X, an X^-, and an electron; it begins at $L = |l_e - l_{X^-}| = 1$ and goes to $L = l_e + l_{X^-} - 1 = 15$.
(iv) A band of multiplicative states containing two neutral excitons and two free electrons; it takes on all even values of L between zero and $2l_e - 1 = 16$.

One interesting feature of figure 13 is that it gives us the effective pseudopotential $V_{AB}(L)$ for the interaction of the pair of fermions AB (where A and B can be e, X^-, X_2^-, etc) as a function of angular momentum. As for electrons, it is convenient to use the relative pair angular momentum $\mathcal{R} = l_A + l_B - L$. For identical fermions with angular momentum l, the allowed values of L are $2l - j$, where j is an odd integer, i.e., $\mathcal{R} = 1, 3, 5, \ldots$, and $\mathcal{R} \leqslant 2l$. For distinguishable fermions A and B, all values of L between $|l_A - l_B|$ and $|l_A + l_B|$ are expected, i.e., $\mathcal{R} = 0, 1, 2, \ldots$, and $\mathcal{R} \leqslant 2\min(l_A, l_B)$. However, our numerical results display a 'hard-core' repulsion for composite particles, and one or more of the pair states with the largest
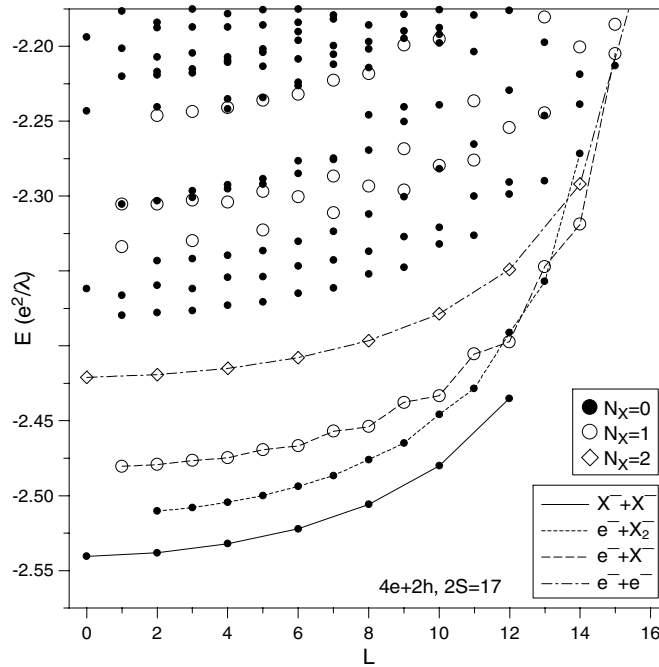

Figure 13. The energy spectrum of a four-electron–two-hole system in the lowest Landau level calculated on a Haldane sphere with $2S = 17$. Different symbols mark states with zero, one, or two decoupled excitons. The lines connect states identified as pseudopotentials of different pairs of bound charged complexes.

values of L (smallest $\mathcal{R}$) are forbidden (i.e. the corresponding pseudopotential parameters are effectively infinite). For $A = X_n^-$ and $B = X_m^-$, the smallest allowed value of $\mathcal{R}$ is given by

$$\mathcal{R}_{AB}^{MIN} = 2\min(n, m) + 1. \tag{27}$$

The identification of pair states AB in figure 13 (as marked with lines) was possible by comparing the displayed 4e–2h spectrum with the pseudopotentials of point charge particles with appropriate angular momenta l_A and l_B and binding energies ε_A and ε_B [19]. The appropriate values of angular momenta l_A and l_B, and of the binding energies ε_A and ε_B are obtained by diagonalizing smaller systems (e.g. the 2e–1h system in figure 12 for an X^-), and the point charge pseudopotentials are used to approximate the AB interaction. The approximate AB energies obtained in this way are rather close to the exact 4e–2h energies. This implies that, due to different energy scales, the internal dynamics of charged excitons is weakly coupled to their scattering off one another or off electrons, and allows for the interpretation of an electron–hole system in terms of well-defined charged excitonic quasiparticles interacting with one another and with excess electrons through Coulomb-like forces. The slight difference between the actual pseudopotentials in figure 13 and the pseudopotentials of point charge particles comes from the larger size of charged excitons and their (nearly frozen) internal degrees of freedom. The latter can be accounted for phenomenologically by attributing to each type of composite particles a finite electric polarizability to describe their induced electric dipole moment in the presence of an electric field of other charged particles. Due to an increased charge isotropy, the polarization effects are expected to be greatly reduced in larger systems, and disappear completely in the fluid-type states discussed in the following paragraphs.

9.3. Generalized composite-fermion picture for charged excitons

Suppose we have a system of different (distinguishable) charged fermions $(A, B, \dots)$. They can be distinguished either because they are different species (e.g., electrons and charged excitons) or because they are confined to different, spatially separated layers. If all particles in such a system repel one another through short-range pseudopotentials (as defined for the electron FQH systems), one can think of many-body states with Laughlin-type correlations [8,9] given by a generalized (compare equation (5)) Laughlin–Jastrow factor

$$\prod (z_i^{(A)} - z_j^{(B)})^{m_{AB}} \tag{28}$$

where $z_i^{(A)}$ is the complex coordinate for the position of the ith fermion of type A, and the product is over all pairs. The restrictions on the integers m_{AB} are that m_{AA} must be odd, $m_{BA} = m_{AB}$, and m_{AB} must not be smaller than certain minimum values $\mathcal{R}_{AB}^{MIN}$ to avoid the infinite hard cores for all pairs. In a state with correlations given by equation (28), a number of pair states with largest repulsion are avoided for each pair, $\mathcal{R}_{AB} \geqslant m_{AB}$. This is equivalent to saying that the high-energy collisions (in which any pair of particles would come very close to one another) are forbidden in such a state. This intuitive property of the Laughlin fluid states will be very useful in the discussion of collision-assisted X^- recombination.

A generalized CF picture can be constructed for a system with Laughlin correlations. In this picture, fictitious flux tubes carrying an integral number of flux quanta ϕ_0 are attached to each particle. In the multi-component system, each particle of type A carries flux $(m_{AA} - 1)\phi_0$ that couples only to charges on all other particles of the same type A and fluxes $m_{AB}\phi_0$ that couple to charges on all particles of other types B (A and B are any of the types of fermions). On a sphere, the effective monopole strength seen by a CF of type A (CF-A) is

$$2S_A^* = 2S - \sum_b (m_{AB} - \delta_{AB})(N_B - \delta_{AB}). \tag{29}$$

For different multi-component systems we expect generalized Laughlin incompressible states (for two components denoted as $[m_{AA}, m_{BB}, m_{AB}]$) when all the hard-core pseudopotentials are avoided and CFs of each kind fill completely an integral number of their CF shells (e.g. $N_A = 2l_A^* + 1$ for the lowest shell). In other cases, the low-lying multiplets are expected to contain different kinds of CF quasiparticles (generalized QEs or QHs), QP-A, QP-B, ..., in the neighbouring incompressible state. It is interesting to realize that the effective monopole strengths $2S_A^*$, i.e. the effective magnetic fields B_A^* seen by particles of different type, are not generally equal. One can think of effective CS charges and fluxes of different colours, but the resulting number of different effective CF magnetic fields of different colour can no longer be regarded as physical reality, and no cancellation between gauge and Coulomb interactions is possible.

The multi-component (multi-colour) CF picture can be applied to electrons and charged excitons in an electron–hole system. We have checked that the pseudopotentials describing interaction of identical composite particles in figure 13 all satisfy the short-range criterion over the entire range of $\mathcal{R}$. For a pair of different particles, the pseudopotential may increase sufficiently quickly for some values of $\mathcal{R}$ but not the others and, for example, for e^- and X^- only the correlations described by exponents $m_{e^- X^-} = 1$ or even, are expected to occur. As an example, let us consider the 12e–6h system. In figure 14 we present its low-energy spectrum at $2S = 17$, calculated by diagonalizing systems of different combinations of electrons and composite particles interacting through effective pseudopotentials determined in figure 13. The following combinations (groupings of 12e and 6h into bound complexes) have the highest total binding energy and thus form the lowest energy bands in the 12e–6h spectrum: (i) $6X^-$,
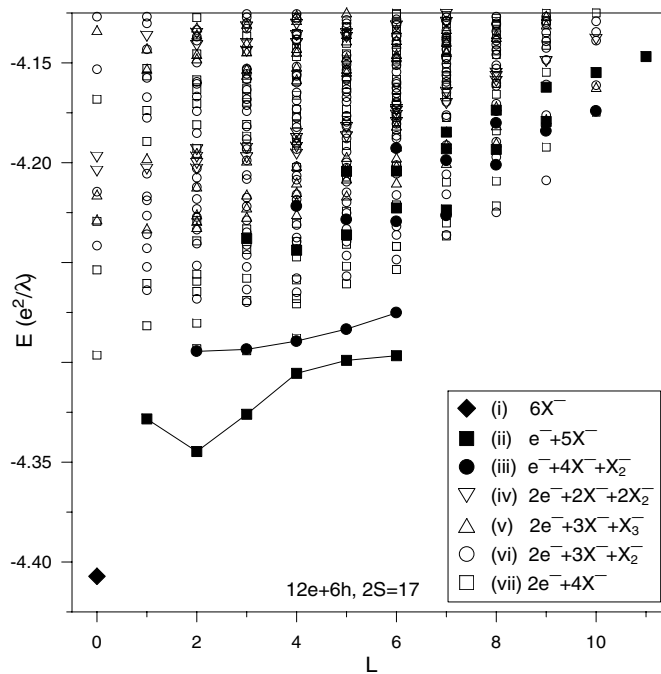

Figure 14. The approximate lowest energy bands corresponding to different combinations of six bound charged complexes interacting through appropriate pseudopotentials, in the twelve-electron–six-hole spectrum in the lowest Landau level, calculated on a Haldane sphere with $2S = 17$. The lines mark lowest subbands of two lowest excited bands.

(ii) $e^- - 5X^-$, (iii) $e^- - 4X^- - X_2^-$, (iv) $2e^- - 2X^- - 2X_2^-$, (v) $2e^- - 3X^- - X_3^-$, (vi) $2e^- - 3X^- - X_2^-$, (vii) $2e^- - 4X^-$. Groupings (ii), (vi), and (vii) also contain neutral excitons that however do not interact with charged particles due to the hidden symmetry. For each of these groupings, the CF transformation can be applied to determine correlations and identify the number and type of quasiparticles that occur in the lowest energy states. For example, for groupings (i)–(iii) the generalized CF picture makes the following predictions:

(i) For $m_{X^- - X^-} = 3$ we obtain the Laughlin $\nu = 1/3$ state with total angular momentum $L = 0$. Because of the hard core of $V_{X^- - X^-}$, this is the only state of this grouping.

(ii) We set $m_{X^- - X^-} = 3$ and $m_{e^- - X^-} = 1, 2,$ and 3. For $m_{e^- - X^-} = 1$ we obtain $L = 1, 2, 3^2,$ $4^2, 5^3, 6^3, 7^3, 8^2, 9^2, 10,$ and 11. For $m_{e^- - X^-} = 2$ we obtain $L = 1, 2, 3, 4, 5,$ and 6. For $m_{e^- - X^-} = 3$ we obtain $L = 1$.

(iii) We set $m_{X^- - X^-} = 3$, $m_{e^- - X_2^-} = 1$, $m_{X^- - X_2^-} = 3$, and $m_{e^- - X^-} = 1, 2,$ or 3. For $m_{e^- - X^-} = 1$ we obtain $L = 2, 3, 4^2, 5^2, 6^3, 7^2, 8^2, 9,$ and 10. For $m_{e^- - X^-} = 2$ we obtain $L = 2, 3, 4, 5,$ and 6. For $m_{e^- - X^-} = 3$ we obtain $L = 2$.

In groupings (ii) and (iii), the sets of multiplets obtained for higher values of $m_{e^- - X^-}$ are subsets of the sets obtained for lower values, and we would expect them to form lower energy bands since they avoid additional small values of $\mathcal{R}_{e^- - X^-}$. However, note that the (ii) and (iii) states predicted for $m_{e^- - X^-} = 3$ (at $L = 1$ and 2, respectively) do not form separate bands in figure 14. This is because $V_{e^- - X^-}$ increases more slowly than linearly as a function of $L(L + 1)$ in the vicinity of $\mathcal{R}_{e^- - X^-} = 3$ (see figure 13). In such a case the CF picture fails [14, 26].

Our conclusion is that different kinds of long-lived fermions (electrons and different charged excitonic complexes) formed in an electron–hole plasma in high magnetic fields can exhibit generalized incompressible FQH ground states with Laughlin-type correlations, and that these states can be described using a generalized CF model.

9.4. Spatially separated electron–hole system

Even in very high magnetic fields (in the lowest Landau level), an asymmetry between e–e, h–h, and e–h interactions can be introduced by spatially separating 2D electron and hole layers. Such separation, which occurs for example in asymmetrically doped wide quantum wells, breaks the hidden symmetry and allows for a rich photoluminescence (PL) spectrum, which (unlike that for a co-planar system) can therefore be used as a probe of the low-lying electron–hole states.

Let us consider an ideal system, in which electrons and holes occupy 2D parallel planes separated by a distance d. The interaction potentials are $V_{ee}(r) = V_{hh}(r) = 1/r$ and $V_{eh}(r) = -1/\sqrt{r^2 + d^2}$. The energy spectrum of a seven-electron–one-hole system is shown in figure 15 for $2S = 15$ and values of d going from 0 to 5 (measured in units of the magnetic length λ). For $d = 5\lambda$, the e–h interaction is weak and, as a first approximation, we can say that the lowest band of states will consist of the lowest CF band of the electron system plus the (constant) hole energy. The allowed angular momenta will be given by L_e, the angular momenta of the low-lying electron states, added to the hole angular momentum l_h of length $l_h = S = 15/2$. At $2S = 15$, the CF picture for the electrons gives $2S^* = 2S - 2p(N - 1) = 15 - 2(7 - 1) = 3$. The seven electrons fill the $l_0^* = 3/2$ shell plus three of the QE states in the shell $l_{QE} = 5/2$. The resulting electron angular momenta are $L_e = 3/2, 5/2,$ and $9/2$. This gives three bands of low-lying states, with total angular momenta $6 \leqslant L \leqslant 9$, $5 \leqslant L \leqslant 10$, and $3 \leqslant L \leqslant 12$, respectively. These three bands can be clearly distinguished at $d = 5\lambda$ and the states within each band become nearly degenerate at $d \sim 10\lambda$.
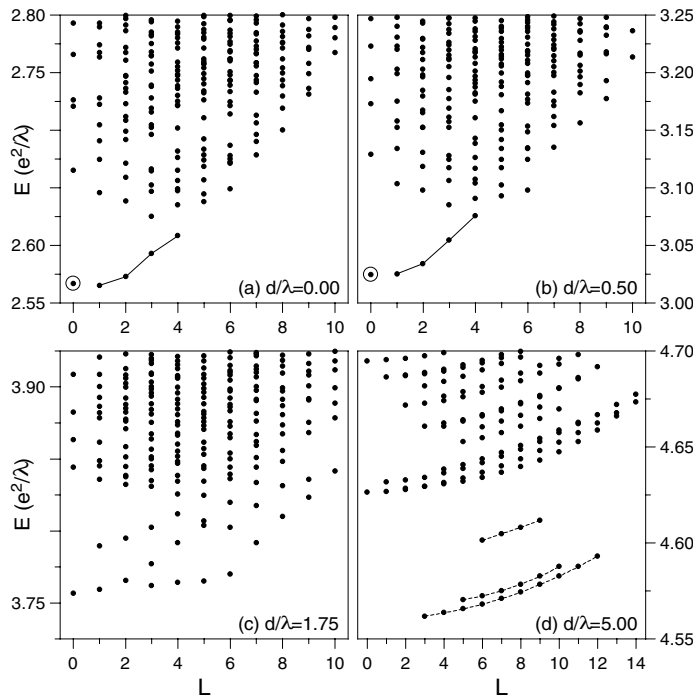
Figure 15. The energy spectrum of a seven-electron–one-hole system in the lowest Landau level calculated on a Haldane sphere at $2S = 15$, for different values of the separation d between electron and hole planes. In frames (a) and (b), the circle marks a multiplicative state and solid lines mark states containing a charged exciton X^-. In frame (d), the dashed lines mark the three lowest bands.

For $d = 0$, it is more useful to consider bound excitonic complexes (X and X^-) and Laughlin quasiparticles of the e^-–X^- fluid. First consider the multiplicative state with a single X and six electrons. At $2S = 15$ six electrons have the Laughlin $\nu = 1/3$ ground state since $2S^* = 15 - 2(6 - 1) = 5$ gives a CF shell which accommodates all six CFs. This is the lowest state at $L = 0$, marked with a circle in frame (a). For a charge configuration containing one X^- and five electrons, we can use the generalized CF model with $m_{e^-e^-} = m_{e^-X^-} = 2$. This gives $2S_e^* = 2S - m_{e^-e^-}(N_e - 1) - m_{e^-X^-} = 5$ and $2S_{X^-}^* = 2S - m_{e^-X^-}N_e = 5$, and the angular momenta $l_e^* = S_e^* = 5/2$ and $l_{X^-}^* = S_{X^-}^* - 1 = 3/2$. There is one empty state in the lowest CF–e^- shell giving $L_e = 5/2$, and the CF–X^- has $L_{X^-} = 3/2$. Adding these two angular momenta gives $L = 1, 2, 3$, and 4 as the lowest band of $5e^-$–X^- states. The multiplicative state at $L = 0$ (open circle) and the band of four multiplets containing an X^- at $L = 1$ to 4 (line) can clearly be seen at $d = 0$ in frame (a). Although the hidden symmetry is only approximate at $d > 0$, these bands can be easily identified at $d = 0.5\lambda$ in frame (b).

At an intermediate separation of $d = 1.75\lambda$ in frame (c), neither description used for $d < \lambda$ or $d \gg \lambda$ is valid, and it seems that a low-energy band occurs at $L = 0, 1, 2, 3^2, 4, 5$, and 6. Most probably, the X^- unbinds but the hole is still able to bind one electron, forming an exciton with a significant electric dipole moment. This dipole moment results in repulsion between the exciton and the remaining six electrons, so the correlations are quite different to those at $d = 0$, where the exciton decouples.

The PL spectrum can be evaluated from the eigenfunctions obtained in the numerical diagonalization of finite systems. For $d \gg 0$, between one and three peaks are observed in the

PL spectrum [25]. Their separations are related to the Laughlin gap (for creation of a QE–QH pair) and to the energy of interaction between the valence band hole and the electron system.

9.5. Charged excitons at a finite magnetic field

One final point is worth mentioning. The numerical calculations described so far were performed for an idealized model in which electrons and holes were confined to infinitely thin 2D layers, and only the lowest Landau level was considered. For realistic systems, effects due to spin, finite width of the quantum well, and Landau-level mixing are very important. The energy spectra of the simple 2e–1h system calculated at $2S = 20$ for parameters appropriate to a 11.5 nm GaAs/AlGaAs quantum well are shown in figure 16. Two frames correspond to the magnetic field of $B = 13$ T and 68 T. We used five electron and hole Landau levels ($n \leqslant 4$) in the calculation, with the realistic magnetic field dependence of the hole cyclotron mass and the appropriate Zeeman splittings. The interaction matrix elements included finite (and different) effective widths of electron and hole quasi-2D layers.
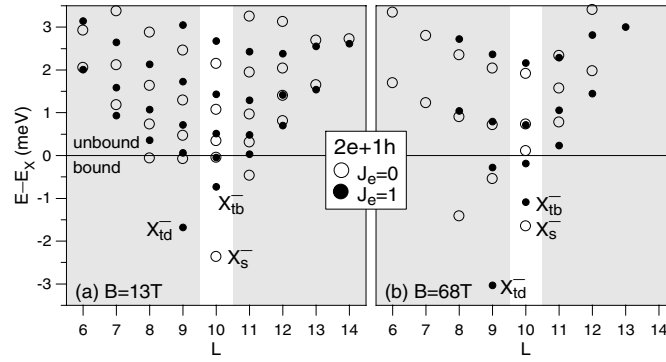

Figure 16. The energy spectra (binding energy versus angular momentum) of the two-electron–one-hole system calculated on a Haldane sphere with the Landau-level degeneracy of $2S + 1 = 21$. Five electron and hole Landau levels are included, and the parameters are appropriate for the 11.5 nm GaAs quantum well in the magnetic field of $B = 13$ T (a) and 68 T (b).

There are a number of bound X^- states in both frames, in contrast to only one singlet bound state at $B = 0$ or only one triplet bound state predicted for an idealized system at infinite B. Three of these bound states are of particular importance. The X_s^- and X_{tb}^- ('b' for 'bright'), the lowest singlet and triplet states at $L = S$, are the only well-bound radiative states, while X_{td}^- ('d' for 'dark') has by far the lowest energy of all non-radiative ($L \neq S$) states. The dark triplet state X_{td}^- is the state discussed in the preceding sections; it is the only bound state in the lowest Landau level, but unbinds at low magnetic fields. The bright singlet state X_s^- is the only bound state at $B = 0$, but unbinds at very high fields due to the hidden symmetry. These states cross at $B \approx 30$ T, as predicted in an earlier calculation [27]. The bright triplet state X_{tb}^- has been discovered very recently [28]. It occurs only at intermediate fields and crosses neither X_s^- nor X_{td}^-. It has larger PL intensity than the X_s^- state.

Although an isolated X_{td}^- is non-radiative because of the angular momentum selection rule, its collisions with other X^-s or with electrons (which break the translational symmetry) could be expected to allow for X_{td}^- recombination. However, the Laughlin correlations limit high-energy collisions at low filling density ($\nu \sim 1/5$ or less) and the PL intensity of a dark X_{td}^- remains very low also in the presence of other particles [28]. In consequence, the X_{td}^- is not seen in PL, and there is no contradiction between experiment [29], which sees recombination

of a triplet state at the energy above the singlet state up to 50 T, and theory [27], which predicts that the lowest triplet state crosses the singlet at roughly 30 T.

10. Summary

We have introduced the Jain CF mean-field picture and shown how the low-lying states can be understood by simple addition of angular momentum. The mean-field CF picture gives the correct spectral structure not because of some cancellation between Chern–Simons and Coulomb interactions beyond the mean-field level, but because it selects a low-angular-momentum subset of the allowed multiplets that avoids the largest pair repulsion. The Laughlin correlations, which describe incompressible quantum fluid states, depend critically on the electron pseudopotential being of 'short range' (by which we mean that $V(L_{12})$ increases more quickly than $L_{12}(L_{12} + 1)$). The validity of Jain's picture also depends upon $V(L_{12})$ being of short range. The pseudopotential describing quasiparticles of a Laughlin condensed state display short-range behaviour only at certain values of L_{12}. We have used this fact to explain why only certain states in the CF hierarchy give rise to incompressible states of the quasiparticle fluid (or daughter states in the hierarchy). The pseudopotentials $V_n(L_{12})$ for higher Landau levels ($n > 0$) do not display short-range behaviour at all values of L_{12}, implying that Laughlin-like correlations will not necessarily result at $\nu' = 2p + \nu$, where p is an integer and ν is a Laughlin–Jain filling factor. The CF ideas have been applied successfully to multi-component plasmas containing different types of fermions with the prediction of possible incompressible fluid states for these systems. Finally, the energy spectrum and PL of electron–hole systems can be interpreted in terms of CFs and Laughlin correlations.

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