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Composite fermions and quantum Hall systems: role of the Coulomb pseudopotential

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ABSTRACT

The mean-field composite fermion (CF) picture successfully predicts angular momenta of multiplets forming the lowest-energy band in fractional quantum Hall (FQH) systems. This success cannot be attributed to a cancellation between Coulomb and Chern–Simons interactions beyond the mean field, because these interactions have totally different energy scales. Rather, it results from the behaviour of the Coulomb pseudopotential $V(L)$ (pair energy as a function of pair angular momentum) in the lowest Landau level (LL). The class of short-range repulsive pseudopotentials is defined that lead to short-range Laughlin-like correlations in many-body systems and to which the CF model can be applied. These Laughlin correlations are described quantitatively using the formalism of fractional parentage. The discussion is illustrated with an analysis of the energy spectra obtained in numerical diagonalization of up to 11 electrons in the lowest and excited LLs. The qualitative difference in the behaviour of $V(L)$ is shown to invalidate sometimes the mean-field CF picture when applied to higher LLs. For example, the $\nu = \frac{7}{3}$ state is not a Laughlin $\nu = \frac{1}{3}$ state in the first excited LL. The analysis of the involved pseudopotentials also explains the success or failure of the CF picture when applied to other systems of charged fermions with Coulomb repulsion, such as the Laughlin quasiparticles in the FQH hierarchy or charged excitons in an electron–hole plasma.

§1. INTRODUCTION

The discovery of the integer quantum Hall (IQH) (von Klitzing *et al*. 1980) and the fractional quantum Hall (FQH) (Tsui *et al*. 1982) effects raised great interest in the properties of a two-dimensional electron gas (2DEG) in high magnetic fields. Both IQH and FQH effects are a manifestation of the occurrence of non-degenerate incompressible ground states in the 2DEG spectrum at certain (integral for IQH and fractional for FQH) Landau level (LL) fillings. However, unlike the single-particle cyclotron gap responsible for the IQH effect, the gap separating a FQH incompressible ground state from the excited states is due to the electron–electron interactions (Laughlin 1983a). While the occurrence of incompressible ground states of both kinds results in quantization of the Hall conductance, the origins of incompressible ground states in the IQH and FQH effects, that is the physics underlying the two quantum Hall effects, are very different.

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A simple picture of the FQH states is offered by the mean-field composite fermion (CF) approach (Jain 1989, Lopez and Fradkin 1991, Halperin *et al.* 1993). The CFs are obtained in the Chern–Simons (CS) gauge transformation, which can be interpreted as attaching to each electron a magnetic flux tube oriented opposite to the external magnetic field B. In the mean-field approximation, the magnetic field of these flux tubes is evenly spread over the occupied area. If the attached flux tubes carry an even number of flux quanta, the CS transformation without the mean-field approximation leaves the energy spectrum and particle statistics unchanged. When the mean-field approximation is made, the effective magnetic field B^* seen by the CFs is lower than the original field B seen by the electrons. The incompressible ground states are predicted to occur at fractional electron fillings that correspond to integer fillings of CF LLs. A gas of strongly interacting electrons is said to behave as a gas of weakly interacting CFs, and the FQH effect of electrons is interpreted as the IQH effect of CFs.

The mean-field CF picture correctly predicts filling factors at which the FQH effect has been experimentally observed. Also, in almost all cases, the mean-field CF predictions of low-lying states of finite systems agree with the results of exact numerical calculations in the lowest LL. However, a very fundamental question, namely 'why does the mean-field CF picture work so well?', is not yet completely understood. The original conjecture that Coulomb and CS gauge interactions beyond the mean-field cancel each other in the FQH systems cannot possibly be correct because the CS interactions are measured on an energy scale proportional to B, which can be much larger than the energy scale of the Coulomb interactions, proportional to $B^{1/2}$. Because so many experimental and numerical results in the lowest LL can be interpreted in terms of CFs, it is extremely important to understand why the CF picture works.

It is known that the CF picture sometimes fails when applied to other systems of identical charged fermions interacting through Coulomb-like forces. For example, the occurrence of incompressible states only at some of the odd denominator fractional filling factors implies that the CF model is not always valid for Laughlin quasiparticles (QPs) in the FQH hierarchy (Haldane 1983, Laughlin 1984, Halperin 1984) or for the CFs themselves in the CF hierarchy (Sitko *et al.* 1996). The CF picture also fails for electrons in the lowest LL, when the layer thickness exceeds certain critical value (Shayegan *et al.* 1990). On the other hand, the numerical experiments show that it is correct for variety of repulsive interaction potentials (e.g. $V(r) \sim -\ln r$ or $r^{-\alpha}$ for $\alpha \geqslant 1$ or even r^{-1}/ϵ with an arbitrary dielectric constant ϵ). The original justification of the CF model rested on the assumption that spontaneously generated gauge interactions cancelled to a substantial extent the repulsive interactions between electrons, independent of the exact form of these interactions. While the CF picture can be used to make certain predictions after it has been established that a certain physical system exhibits incompressible fluid ground states with Laughlin-like correlations at appropriate conditions (magnetic field, electron density, layer thickness, disorder, material parameters, etc.), it cannot predict its own validity for such a system. Therefore, another very fundamental problem, namely 'when does the mean-field CF picture work?', needs to be answered.

In this paper, we explain the connection between the form of the Coulomb pseudopotential (Haldane 1987) $V(L)$, defined as the dependence of the pair interaction energy V on the pair angular momentum L, and the occurrence of the incompressible ground states in the lowest LL of an interacting 2DEG at Laughlin–Jain filling factors $\nu = \frac{1}{3}, \frac{1}{5}, \frac{2}{5}, \frac{2}{7}$, etc. We present arguments justifying the validity of the

mean-field CF picture for the lowest LL and show when and why it can be used. It is known that the electrons in Laughlin $\nu = (2p+1)^{-1}$ states avoid a number p of pair states with largest repulsion (Haldane 1987). The origin of incompressible FQH states at certain other filling factors, such as $\nu = \frac{2}{5}$, has been also attributed to the ability of avoiding strongly repulsive pair states (Halperin 1983, Haldane 1987, Rezayi and MacDonald 1991, Belkhir and Jain 1993). In order to treat formally the ability to avoid certain pseudopotential parameters in the incompressible many-body states we use the formalism of fractional parentage, well established in nuclear (de Shalit and Talmi 1963) and atomic (Cowan 1981) physics. It is shown that the condition for the validity of the mean-field CF picture can be more easily expressed in terms of the behaviour of the pseudopotential $V(L)$ than in terms of the behaviour of the interaction potential $V(r)$. The condition on the form of interaction pseudopotential necessary for the occurrence of FQH states is given, which defines the class of short-range pseudopotentials to which the mean-field CF picture can be applied. It is shown that the Coulomb interaction in the lowest LL falls in this class, while in higher LLs the mean-field CF picture can be used only below a certain filling factor. Similarly, the success or failure of the mean-field CF picture applied to Laughlin QPs, depending on the type of QPs and their filling factor (Sitko *et al.* 1997), is shown to reflect the behaviour of appropriate QP pseudopotentials. It is argued that a QP hierarchy picture taking into account the qualitative features of involved pseudopotentials (Wójs and Quinn 2000) should most naturally explain the occurrence and relative stability of observed odd denominator FQH states. We are not discussing even denominator fractions (Willet *et al.* 1987), which are explained in terms of pairing of electrons (Haldane and Rezayi 1988, Moore and Read 1991), although a pseudopotential approach to the interaction between bound pairs might be possible. The discussion throughout the paper is illustrated by exact numerical calculations of energy spectra and parentage coefficients in Haldane's (1983) spherical geometry for up to 11 electrons at $\nu \approx \frac{1}{3}$ and up to eight electrons at $\nu \approx \frac{1}{5}$ in the lowest and excited LLs (matrix dimensions up to 3×10^6), using a modified Lanczos (1950) algorithm (Haydock 1980).

The paper is organized as follows. In § 2 a brief overview of the numerical (exact diagonalization) calculations on the Haldane sphere is given. In § 3 the mean-field CF picture of the FQH states is explained. The success of the mean-field CF approach is illustrated in the energy spectra of the eight-electron system in the lowest LL, for filling factors between $\nu = \frac{1}{3}$ and $\frac{1}{5}$. In § 4 the interaction pseudopotential is introduced. In § 5 the three-electron system is discussed. The idea of fractional parentage from pair states is used to characterize the three particle states. The energy spectra in the lowest and excited LLs are analysed and interpreted in terms of pseudopotential and fractional parentage. In § 6 the analysis of the three-electron case is generated to an arbitrary electron number, and the numerical results for up to 11 electrons are presented. In § 7 the relation between the form of the interaction pseudopotential and the occurrence of many-electron incompressible ground states is explained. The Coulomb interaction in different LLs is compared with the harmonic repulsive interaction and the Coulomb interaction in the atomic shells. Hund's rule appropriate for FQH systems is formulated. The short-range pseudopotential is defined, to which the CF model can be applied. The prescription for the low-energy many-electron multiplets is derived, which agrees with predictions of the mean-field CF picture. The consequences of the form of pseudopotential for condensation of QPs in the hierarchy picture are mentioned. The conclusions are contained in § 8.

§ 2. Numerical studies

2.1. *Introduction*

In a magnetic field B, the lowest LL of a 2DEG can accommodate $N_\phi = BC/\phi_0$ electrons per area C ($\phi_0 = hc/e$ is the magnetic flux quantum). The measure of electron density is the fraction of occupied states, given by the filling factor $\nu = N/N_\phi$, where N is the number of electrons in the area C. In the absence of electron–electron interactions, the N_ϕ single-particle states are degenerate. Therefore, these interactions entirely determine the low-energy spectrum of the system at $\nu < 1$ and cannot be treated perturbatively. Instead, numerical diagonalization techniques have commonly been employed, which, however, limit the system to a finite (small) number of electrons. Different approaches restrict the motion of a finite number of electrons to a finite area C to model an infinite 2DEG at a finite density include imposing a lateral (parabolic, hard wall, etc.) confinement (Laughlin 1983a), using periodic boundary conditions (Haldane and Rezayi 1985b), or confining electrons on a closed surface (Haldane 1983). The last approach has proven particularly useful, since it naturally avoids edge effects. Also, the translational symmetry of a (planar) 2DEG is preserved in the form of the rotational symmetry of a sphere. In particular, the pair of good quantum numbers resulting from the translational symmetry of a plane, namely the centre of mass and relative momenta, correspond to the pair of good quantum numbers on a sphere, namely the total angular momentum L and its projection L_z (Wójs and Quinn 1998a). Consequently, the degeneracies associated with centre-of-mass excitations on a plane correspond to those associated with different values of L_z ($|L_z| \leqslant L$) on a sphere, and the non-degenerate incompressible ground states of a planar 2DEG correspond to non-degenerate ($L = 0$) ground states on a sphere.

2.2. *Haldane sphere*

The magnetic field B perpendicular to the surface of the Haldane sphere of radius R is an isotropic radial field produced by a magnetic monopole placed at the origin. The monopole strength $2S$ is defined in the units of elementary flux $\phi_0 = hc/e$, so that the total flux through the sphere is $4\pi BR^2 = 2S\phi_0$. Dirac's (1931) monopole quantization condition requires that $2S$ is an integer, and positive S means the magnetic field pointing outwards. The convenient units of length and energy, magnetic length λ and the cyclotron frequency $\hbar\omega_c$, are given by

$$\lambda^2|S| = R^2, \tag{1}$$

$$\hbar\omega_c = S\,\frac{\hbar^2}{\mu R^2}. \tag{2}$$

The eigenstates of the single-particle Hamiltonian are denoted by $|S,l,m\rangle$ and called monopole harmonics (Wu and Yang 1976). They are labelled by the angular momentum l and its projection m. The degenerate angular momentum shells are equivalent to the LLs of the planar geometry. The eigenenergies are given by

$$\begin{aligned} E_n &= \frac{\hbar\omega_c}{2S}\left[l(l+1) - S^2\right] \\ &= \hbar\omega_c\left[n + \frac{1}{2} + \frac{n(n+1)}{2S}\right], \end{aligned} \tag{3}$$

where the shell (LL) index is defined as $n = l - S = 0,1,2,\ldots$. The degeneracy of each shell (LL) is $N_\phi = 2l + 1$.

For the FQH states at filling factors $\nu < 1$, only the lowest spin polarized shell (LL) need be considered. It corresponds to $n = 0$ ($l = S$), and for simplicity its single-particle states will be denoted as $|m\rangle$. The spin polarized FQH states in excited LLs will also be studied. Owing to the high (cyclotron) energy of the inter-LL excitations in high magnetic fields, the FQH states at filling factors $2n < \nu < 2n+1$ are composed of completely filled LLs (spin up and down) up to the $(n-1)$th LL, and a partially filled nth LL with the filling factor $\nu_n < 1$ (we discuss only the spin-polarized states in partially filled excited LLs). The Hartree–Fock energy describing interaction between an electron in the nth LL and the underlying completely filled LLs is a constant. Therefore, the energy spectrum of N electrons at $\nu_n < 1$ in an isolated nth LL describes (up to this constant) the low-energy spectrum of $N + 2n(2S+n)$ electrons at $\nu = 2n + \nu_n$. Since states of only one LL with a given n appear in the 'reduced' problem for $\nu = 2n + \nu_n$, the following simplified notation will be used: the filling factor ν_n will be denoted as ν, and the states $|S,l,m\rangle$ will be denoted as $|m\rangle$.

2.3. *Many-body problem*

The object of numerical studies is to diagonalize the electron–electron interaction Hamiltonian

$$\hat{H} = \sum_{m_1 m_2 m_3 m_4} c^\dagger_{m_1} c^\dagger_{m_2} c_{m_3} c_{m_4} \langle m_1, m_2 | V | m_3, m_4 \rangle \tag{4}$$

within the Hilbert space $\mathcal{H}_{\mathrm{MB}}$ of $N_{\mathrm{MB}} = N_\phi! [N!(N_\phi - N)!]^{-1}$ degenerate antisymmetric N electron states of a given (N_ϕ-fold degenerate) LL. In the above, $c^\dagger_m$ (c_m) creates (annihilates) an electron in the state $|m\rangle$. The two-body Coulomb matrix elements have a particularly simple form in the lowest LL (Fano *et al.* 1986), but they can also be evaluated analytically for a general case of inter- or intra-LL scattering. The N-electron Hilbert space $\mathcal{H}_{\mathrm{MB}}$ is spanned by single-particle configurations $|m_1, m_2, \ldots, m_N\rangle$, classified by the total angular momentum projection $M = m_1 + m_2 + \cdots + m_N$. Taking advantage of the Wigner–Eckart theorem, each (M) subspace $\mathcal{H}_{\mathrm{MB}}(M)$ can be further block diagonalized into (M,L) subspaces $\mathcal{H}_{\mathrm{MB}}(M,L)$ corresponding to different values of the total angular momentum L. The Wigner–Eckart theorem tells us that, because the interaction Hamiltonian is a scalar, its matrix element between angular momentum eigenstates $|L,M,\alpha\rangle$ can be written as

$$\langle L', M', \alpha' | \hat{H} | L, M, \alpha \rangle = \delta_{LL'} \delta_{MM'} V_{\alpha\alpha'}(L), \tag{5}$$

that is in terms of a reduced matrix element

$$V_{\alpha\alpha'}(L) = \langle L, \alpha' | \hat{H} | L, \alpha \rangle, \tag{6}$$

which is independent of M. Here, the index α distinguishes different states in the same space $\mathcal{H}_{\mathrm{MB}}(M,L)$. Typical dimensions are given in table 1, where we list the dimension of the total Hilbert space $\mathcal{H}_{\mathrm{MB}}$, of the largest (M) subspace $\mathcal{H}_{\mathrm{MB}}(0)$, of the largest (M,L) subspace $\mathcal{H}^{\max}_{\mathrm{MB}}(M,L)$ and of the (M,L) subspace containing the Laughlin $L = 0$ ground state, $\mathcal{H}_{\mathrm{MB}}(0,0)$, for between six and 11 electrons at the filling factor $\nu = \frac{1}{3}$. Even when both M and L are resolved, exact diagonalization becomes difficult when $N > 10$ and $N_\phi > 28$.

Table 1. The dimensions N_ϕ of the single-particle Landau level, dimensions N_{MB} of the total many body Hilbert space $\mathcal{H}_{\mathrm{MB}}$, dimensions $N_{\mathrm{MB}}(0)$ of the largest (M) subspace $\mathcal{H}_{\mathrm{MB}}(0)$, dimensions $N_{\mathrm{MB}}^{\max}(M,L)$ of the largest (M,L) subspace $\mathcal{H}_{\mathrm{MB}}^{\max}(M,L)$, and dimensions $N_{\mathrm{MB}}(0,0)$ of the (M,L) subspace $\mathcal{H}_{\mathrm{MB}}(0,0)$ containing the Laughlin $L=0$ ground state of $N=6$–11 electrons at the filling factor $\nu=\frac{1}{3}$.

N	N_ϕ	N_{MB}	$N_{\mathrm{MB}}(0)$	$N_{\mathrm{MB}}^{\max}(M,L)$	$N_{\mathrm{MB}}(0,0)$
6	16	8 008	338	24	6
7	19	50 388	1 656	86	10
8	22	319 770	8 512	352	31
9	25	2 042 975	45 207	1 533	84
10	28	13 123 110	246 448	7 069	319
11	31	84 672 315	1 371 535	33 787	1 160

The calculations give the eigenenergies E as a function of the total angular momentum L. The numerical results for the lowest LL always show one or more L multiplets forming a low-energy sector (or low-energy band). The spectra for N in the range 6–20 (depending on the filling factor) are available in literature and have been extensively analysed. As an example, in figures 1 and 2 we show the energy spectra obtained for eight electrons in the lowest LL, at values of $2S$ between 21 and 37; the spectra for $2S<21$ can be found in earlier numerical studies (He *et al.* 1992). The Laughlin filling factors $\nu=\frac{1}{3}$ and $\frac{1}{5}$ occur at $2S=21$ and 35, and the Jain filling factors $\nu=\frac{2}{7}$ and $\frac{2}{9}$ occur at $2S=26$ and 30 respectively. At $2S=28$, an even denominator filling of $\nu=\frac{1}{4}$ occurs. The low-energy bands are indicated by open circles. For some values of $2S$ these bands contain subbands indicated by broken lines. The physical meaning of the bands indicated in figures 1 and 2 will be explained in § 3.2

§ 3. Composite fermion approach

3.1. *Introduction*

In the (CS) transformation, an equal and even number $(2p)$ of elementary fluxes ϕ_0 (a fictitious flux tube of strength $2p\phi_0$) oriented opposite to the original magnetic field B is attached to each electron. The CFs obtained in this way carry electric charge and magnetic flux. The CS transformation is a gauge transformation and thus the CF energy spectrum is identical with the original electron spectrum.

Since attached fluxes are localized on electrons and the magnetic field acting on each electron is unchanged, the classical Hamiltonian of the system is also unchanged. However, the quantum Hamiltonian includes additional terms describing an additional charge–flux (CS) interaction, which arises from the Aharonov–Bohm phase attained when one electron's path encircles the flux tube attached to another electron. One difficulty in treatment of the CS interaction results from the fact that it contains both two- and three-body terms; another is the absence of a small parameter with which to construct a perturbation expansion.

3.2. *Mean-field approximation*

In the mean-field approach, the magnetic field due to attached flux tubes is evenly spread over the occupied area C. The mean-field CFs obtained in this way move in an effective magnetic field $B^*=B-2p\phi_0 N/C$. An effective filling factor ν^* seen by one CF is defined as

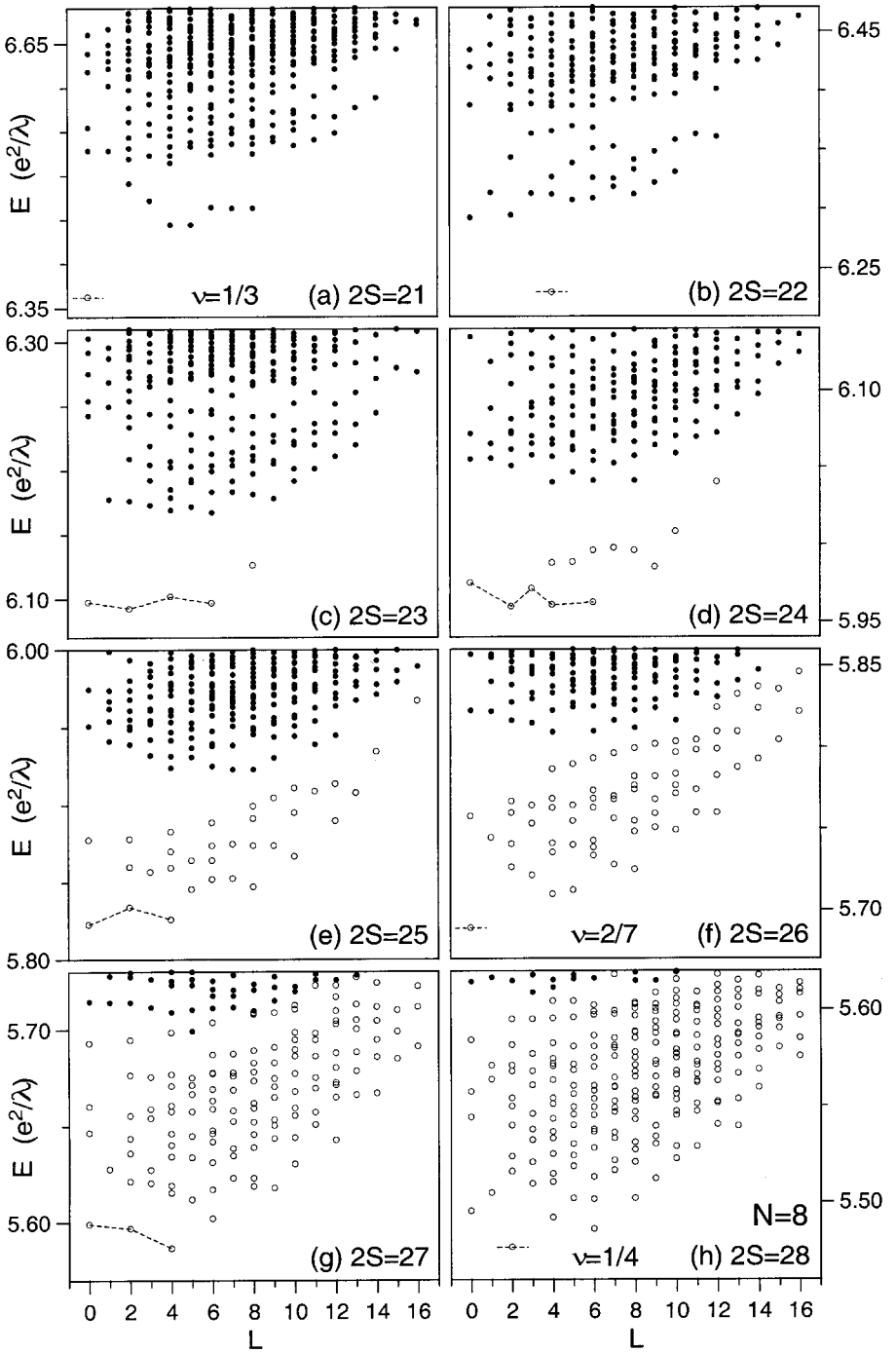

Figure 1. The energy spectra of eight electrons in the lowest Landau level at the monopole strength $2S$ between 21 and 28: (*a*) $2S = 21$ corresponds to the filling factor $\nu = \frac{1}{3}$, the lowest-energy state at $L = 0$ is the Laughlin ground state; (*b*) $2S = 22$; (*c*) $2S = 23$; (*d*) $2S = 24$; (*e*) $2S = 25$; (*f*) $2S = 26$, $\nu = \frac{2}{7}$, Jain ground state at $L = 0$; (*g*) $2S = 27$; (*h*) $2S = 28$, $\nu = \frac{1}{4}$. The low-energy states selected by the CS transformation with $p = 1$ and $p = 2$ are indicated by open circles and broken lines, respectively.

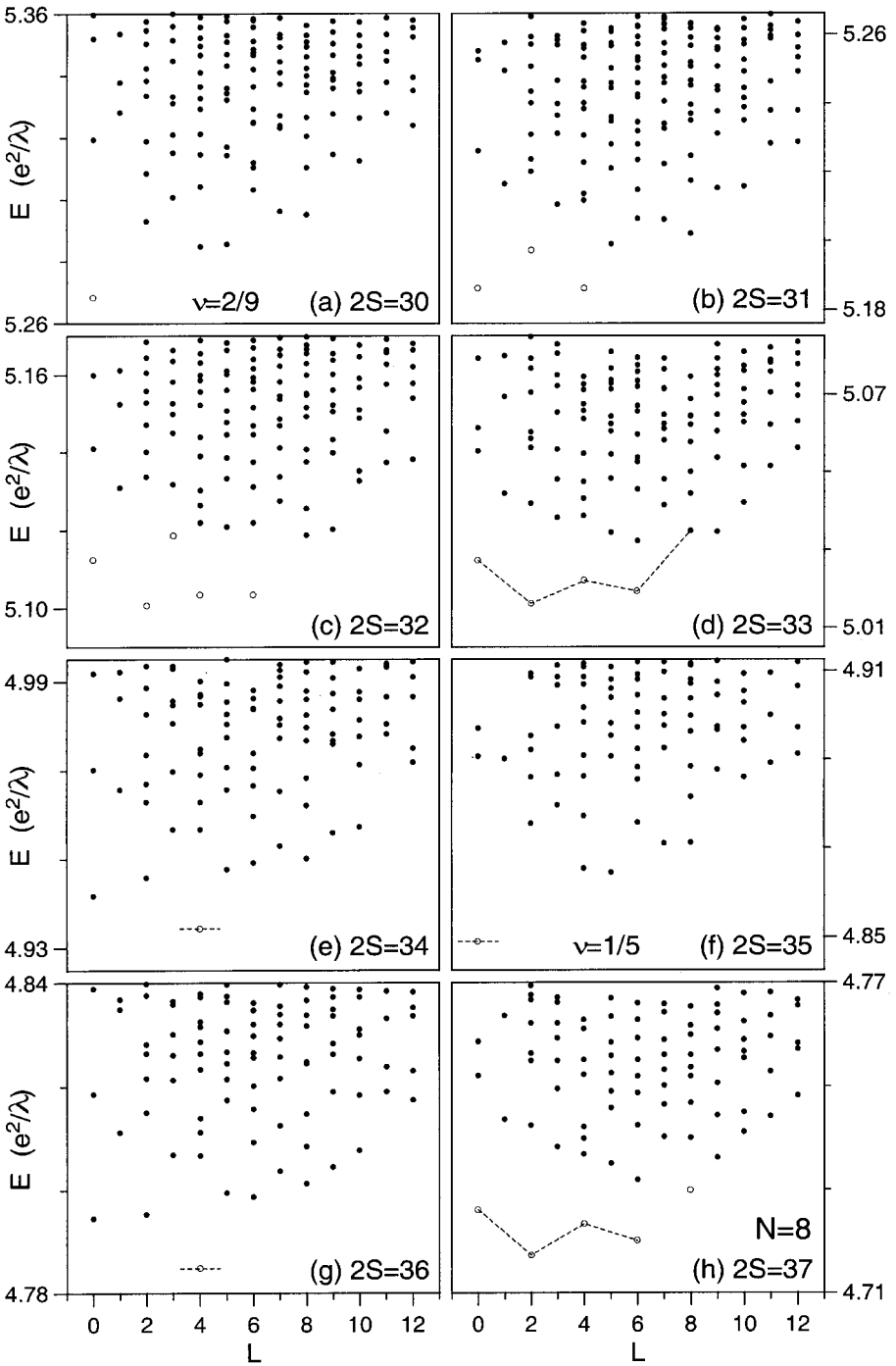

Figure 2. The energy spectra of eight electrons in the lowest Landau level at the monopole strength $2S$ between 30 and 37; (*a*) $2S = 30$ corresponds to the filling factor $\nu = \frac{2}{9}$, the lowest-energy state at $L = 0$ is the Jain ground state; (*b*) $2S = 31$; (*c*) $2S = 32$; (*d*) $2S = 33$; (*e*) $2S = 34$; (*f*) $2S = 35$, $\nu = \frac{1}{5}$, Laughlin ground state at $L = 0$; (*g*) $2S = 36$; (*h*) $2S = 37$. The low-energy states selected by the CS transformation with $p = 2$ and $p = 3$ are indicated by open circles and broken lines, respectively.

$$(\nu^*)^{-1} = \nu^{-1} - 2p, \tag{7}$$

so that

$$B^*\nu^* = B\nu = \frac{N}{C}\,\phi_0. \tag{8}$$

Negative ν^* means negative B^* (oriented opposite to B). It has been shown that the mean-field Hamiltonian of non-interacting CFs gives a good qualitative description of the low-lying states of interacting electrons in the lowest LL. The Jain (1989) sequence of incompressible ground states is predicted at filling factors ν for which ν^* is an integer, and the $\nu^* = 1$ states correspond to Laughlin (1983a) $\nu = (2p+1)^{-1}$ states. If ν^* is not an integer, the low-lying states contain a number of QPs ($N_{\rm QP} \leqslant N$) in the neighbouring incompressible state with integer ν^*.

On a sphere, an effective CF monopole strength is

$$2S^* = 2S - 2p(N-1), \tag{9}$$

and $l^* = |S^*|$ plays the role of the angular momentum of the lowest CF shell (Chen and Quinn 1996). If n lowest CF LLs at $2S^*$ are filled completely by N CFs, the corresponding N-electron state at $2S$ is incompressible. The states at other values of $2S$ are compressible and contain $N_{\rm QP}$ QPs in the neighbouring incompressible state of an equal number of N electrons at $2S_{\rm inc}$,

$$N_{\rm QP} = n(|2S^*_{\rm inc}| - |2S^*|). \tag{10}$$

Here $2S^*_{\rm inc}$ is the effective monopole strength calculated for the incompressible state, that is $2S^*_{\rm inc} = 2S_{\rm inc} - 2p_{\rm inc}(N-1)$, and n is an integral number of completely filled CF LLs. Positive $N_{\rm QP}$ corresponds to quasielectrons (QEs) in the $(n+1)$th (lowest unoccupied) CF shell, each with angular momentum $l_{\rm QE} = l^* + n$. Negative $N_{\rm QP}$ corresponds to quasiholes (QHs) in the nth (highest occupied) CF shell, each with angular momentum $l_{\rm QH} = l^* + n - 1$. Different values of $2S$ that lead to the same value of $l^* = |S^*|$ are equivalent and their low-energy bands contain the same L multiplets.

It is noteworthy that the CS transformation applied to the state at $2S$ can have a different flux strength ($2p$) than that ($2p_{\rm inc}$) applied to the incompressible state $2S_{\rm inc}$. Consequently, alternative pictures of the $(N, 2S)$ state, containing different numbers of QPs and/or types of QP, can be obtained (Yi *et al.* 1996). Writing $p_{\rm inc}$ and p explicitly, equation (10) can be written as

$$N_{\rm QP} = n(|2S_{\rm inc} - 2p_{\rm inc}(N-1)| - |2S - 2p(N-1)|). \tag{11}$$

The original spectrum of interacting electrons is similar to that of non-interacting mean-field CFs in a sense that, firstly, the lowest band of angular momentum multiplets contains states of the minimum number of QPs consistent with the values of N and $2S$ and, secondly, neighbouring excited bands contain additional QE–QH pairs.

Let us illustrate the success of the mean-field CF approach in predicting the lowest band of multiplets in the example of an eight-electron system. The sequence of incompressible states is given in table 2. Eight mean-field CFs fill completely one CF LL ($n = 1$) at $|2S^*| = 7$ and two CF LLs ($n = 2$) at $|2S^*| = 2$. Following equation (9), the sequences of incompressible states for CF fillings $n = 1$ and 2 are generated by varying $p = 0, \pm 1, \pm 2, \ldots$. States listed in table 2 ($\nu = 2, 1, \frac{2}{3}, \frac{2}{5}, \frac{1}{3}, \frac{2}{7}, \frac{2}{9}$, and $\frac{1}{5}$) are all the incompressible eight-electron states at filling factors greater than or equal to $\nu = \frac{1}{5}$ (filling of more than two CF LLs requires $N > 8$). The states

Table 2. The incompressible states of eight electrons; filling factor $\nu \geqslant \frac{1}{5}$.

	$n=1, 2S^*=7$		$n=2, 2S^*=2$	
p	$\|2S\|$	ν	$\|2S\|$	ν
-2	21	$\frac{1}{3}$	26	$\frac{2}{7}$
-1	7	1	12	$\frac{2}{3}$
0	7	1	2	2
1	21	$\frac{1}{3}$	16	$\frac{2}{5}$
2	35	$\frac{1}{5}$	30	$\frac{2}{9}$

outside the incompressible sequence of $2S_{\text{inc}} = 2,7,12,16,21,26,30,35,\ldots$ are compressible and contain an appropriate number of QPs, given by equation (10).

The spectra of an eight-electron system in the lowest LL for values of $2S$ between 21 and 37, that is for the filling factors ν from $\frac{1}{3}$ down to below $\frac{1}{5}$, are shown in figures 1 and 2. In figure 1, the open circles and broken lines indicate bands of multiplets predicted in the mean-field CF picture as the lowest-energy states of CFs for $p=1$ and $p=2$, respectively. In figure 2, all shown states belong to the lowest band corresponding to $p=1$, and the open circles and broken lines indicate bands obtained for $p=2$ and $p=3$ respectively. The range of $2S$ shown in figure 1 alone covers all values of l^* from $N-1$ to 0 (for $p=1$) and thus exhausts all possible configurations of QPs for the eight-electron system. Let us analyse the spectra in figure 1 in greater detail.

At $2S=21$ the CS transformation with $p=1$ gives $2S^*=7$. The lowest CF LL is completely filled ($\nu^*=n=1$) and the Laughlin incompressible $\nu_{\text{inc}}=\frac{1}{3}$ state with $L=0$ is formed. The CS transformation with $p=2$ gives $2S^*=-7$ and the equivalent interpretation of the ground state. At $2S=22$ the CS transformation with $p=1$ gives $2S^*=8$. The lowest CF LL has degeneracy of $2S^*+1=9$; so it holds $N=8$ CFs and one QH with $l_{\text{QH}}=4$ (QH in the $\nu_{\text{inc}}=\frac{1}{3}$ state). Therefore, the low-energy band contains a single multiplet with $L=4$. The CS transformation with $p=2$ gives $2S^*=-6$, which corresponds to a completely filled lowest CF LL and one QE with $l_{\text{QE}}=4$ in the first excited CF LL. Depending on the applied CS transformation, the $L=4$ ground state can be viewed as a state of either a single QE or a single QH in the appropriate CF LL (Yi *et al.* 1996). The low-energy multiplets obtained using the CS transformation with $p=1$ at $2S=23,24,\ldots,28$ contain $2,3,\ldots,7$ QHs in the lowest CF LL (i.e in the $\nu_{\text{inc}}=\frac{1}{3}$ state), each with angular momentum $l_{\text{QH}}=\frac{9}{2},5,\ldots,7$ respectively. For example, at $2S=24$ the band of states of three QHs each with $l_{\text{QH}}=5$ contains the following multiplets: $L=0,2,3,4^2,5,6^2,7,8,9,10$ and 12. At $2S\geqslant 23$ the CS transformation with $p=2$ selects a subset of multiplets out of those obtained with $p=1$, and the low-energy subband corresponding to $p=2$ develops in the $p=1$ band. For example, the low-energy $p=2$ subband predicted for $2S=23$ ($2S^*=-5$) contains two QEs each with $l_{\text{QE}}=\frac{7}{2}$, and thus $L=0$, 2, 4 and 6. At $2S=26$ the CF monopole strength for $p=2$ is $2S^*=-2$ and the two lowest CF LLs are completely filled ($\nu^*=n=2$). The ground state is the incompressible Jain $\nu_{\text{inc}}=\frac{2}{7}$ state with $L=0$. At $2S=25$ the CF monopole strength for $p=2$ is $2S^*=-3$ and at $2S=27$ it is $2S^*=-1$. In both cases, the low-energy band contains two QPs each with $l_{\text{QP}}=\frac{5}{2}$ in the $\nu_{\text{inc}}=\frac{2}{7}$ state

(two QEs at $2S = 25$ and two QHs at $2S = 27$). For $2S = 24$ one obtains $2S^* = -4$ and the lowest-energy band contains three QEs, each with $l_{QE} = 3$. Finally, for $2S = 28$, one obtains $2S^* = 0$ and one QH with $l_{QH} = 2$ in the second excited CF LL. The effective magnetic field acting on the CFs vanishes, and this state is assigned an even denominator filling factor $\nu = \frac{1}{4}$.

Higher-energy bands, containing multiplets with additional QE–QH pairs, are more difficult to identify in figures 1 and 2 than the lowest bands. However, for $2S = 21$, one can easily see the low-lying band of states at $L = 2$, 3, 4, 5, 6, 7 and 8, which correspond to the states of one QE–QH pair ($l_{QE} = \frac{9}{2}$ and $l_{QH} = \frac{7}{2}$) in the mean-field CF picture. Similarly, the band of QE–QH pair states for $2S = 26$ occurs at $L = 2$, 3, 4, and 5 ($l_{QE} = 3$ and $l_{QH} = 2$). For $2S = 25$ the lowest band contains two QHs, each with $l_{QH} = \frac{5}{2}$, in the $\nu^* = 2$ CF state ($L = 0$, 2, and 4). The first excited band has two subbands at the same CF energy. One contains states corresponding to three QHs, each with $l_{QH} = \frac{5}{2}$, and one QE with $l_{QE} = \frac{7}{2}$. The allowed multiplets of such QP system are $L = 1^2$, 2^3, 3^3, 4^3, 5^3, 6^2, 7, and 8. The other contains states of one QH in the lowest CF LL ($l_{QH1} = \frac{3}{2}$) and one QH in the first excited CF LL ($l_{QH2} = \frac{5}{2}$). The allowed multiplets in this subband are $L = 1$, 2, 3 and 4. One can identify in figure 1 (*f*) a few multiplets with highest angular momenta ($L = 8$, 7, $6^2, \ldots$) of this band.

The bands of states containing an increasing number of QE–QH pairs are more clearly visible in the density of states (DOS), $\mathrm{d}\mathcal{N}(E)/\mathrm{d}E$, plotted in figure 3. Figures 3 (*a*) and (*b*) show the data for $2S = 21$ (Laughlin $\nu = \frac{1}{3}$ ground state) and $2S = 22$ (one QH in the ground state) respectively. The continuous DOS is obtained by broadening of discrete energy levels with Gaussians:

$$\frac{\mathrm{d}\mathcal{N}(E)}{\mathrm{d}E} = \frac{\pi^{1/2}}{\delta} \sum_{L\alpha} (2L+1) \exp\left(-\frac{|E - E_{L\alpha}|^2}{\delta^2}\right) \tag{12}$$

where the summation goes over all L multiplets (distinguished by different α), and the normalization pre-factor guarantees that $\int [\mathrm{d}\mathcal{N}(E)/\mathrm{d}E]\, \mathrm{d}E = \mathcal{N}$, the total number of states. The thin lines were obtained for $\delta = 0.001\, e^2/\lambda$ and the thick lines correspond to $\delta = 0.02\, e^2/\lambda$. The thick lines, free of noise characteristic of the discrete spectrum, reveal a series of equidistant peaks and/or steps in the DOS. The peaks corresponding to the ground states are hardly visible and their positions have been indicated by open circles. A number of higher peaks (at lower energies) or plateaux (at higher energies) are the remnants of the CF bands with increasing numbers of QPs. The quasiperiodic character of the DOS spectrum is even more pronounced in the derivatives of the DOS, shown in the insets (calculated only for $\delta = 0.02\, e^2/\lambda$). The plateaux in $\mathrm{d}\mathcal{N}\,\mathrm{d}E$ correspond to the minima in $\mathrm{d}^2\mathcal{N}/\mathrm{d}E^2$, and the average distance between the neighbouring minima is about $0.094\, e^2/\lambda$. In the mean-field CF picture, this quantity is interpreted as the energy of a QE–QH pair in the Laughlin $\nu = \frac{1}{3}$ ground state.

The Fermi-liquid picture can be further applied to the QPs (Sitko *et al.* 1996). The incompressible state is treated as a 'vacuum' state, and the QPs created in this state interact with one another through appropriate pseudopotentials. The pseudopotentials were determined by studying the energy spectra corresponding to two QPs, and then used to calculate the QP–QP interaction energy in states corresponding to a larger number of QPs. Good agreement with the actual low-energy bands of the electron systems was obtained.

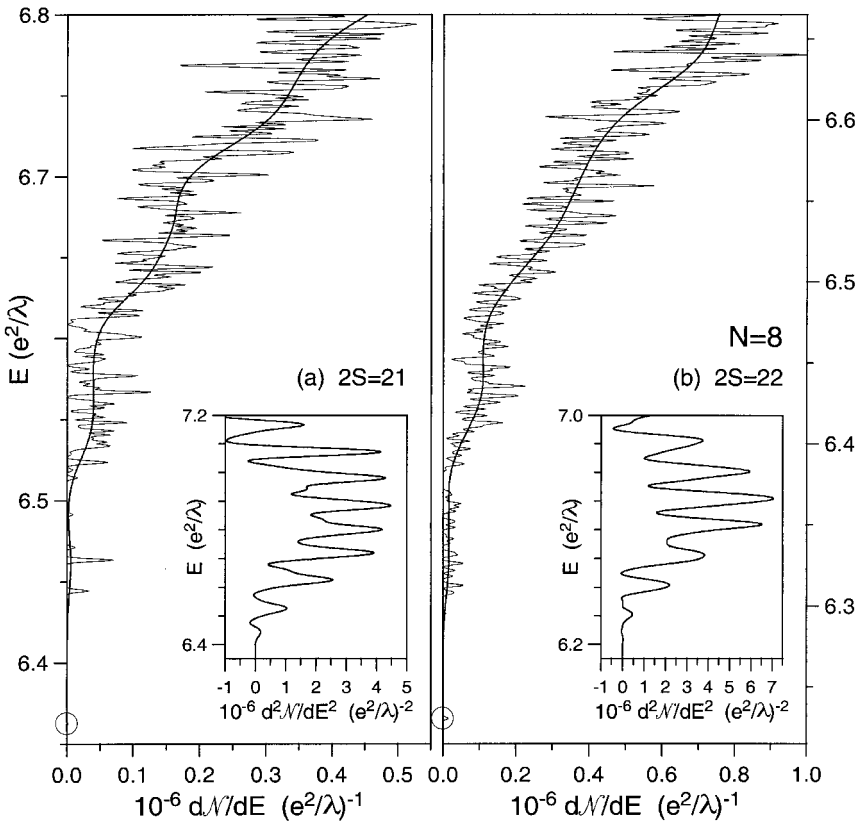

Figure 3. The DOS $\mathrm{d}\mathcal{N}/\mathrm{d}E$ for the eight-electron spectra at (*a*) $2S = 21$ and (*b*) $2S = 22$. The thin and thick lines correspond to two different broadenings of discrete energy levels. The inset shows the differential DOS $\mathrm{d}^2\mathcal{N}/\mathrm{d}E^2$. The plateaux in $\mathrm{d}\mathcal{N}/\mathrm{d}E$ and the minima in $\mathrm{d}^2\mathcal{N}/\mathrm{d}E^2$ correspond to the bands of states with an increasing number of QE–QH pairs.

3.3. *Energy scales and fluctuations beyond the mean-field approximation*

Despite the success of the mean-field CF approach in describing the low-energy spectra of interacting electrons in many numerical (exact) calculations carried out for finite systems, the reason for its success still remains a puzzle. The original conjecture that the CF transformation converts a system of strongly interacting electrons into one of weakly interacting CFs cannot possibly be correct because the CS interactions among fluctuations are measured on an energy scale proportional to $\hbar\omega_c \propto B$, which can be much larger than the energy scale of the Coulomb interactions, proportional to $e^2/\lambda \propto B^{1/2}$. This is demonstrated in figure 4, where the original energy spectrum of free electrons is compared with that of non-interacting mean-field CFs (note that the degeneracy of multiplets is not shown). Clearly, inclusion of the electron–electron Coulomb interaction with characteristic energy as small as indicated in figure 4 by a grey rectangle cannot reproduce the separation of levels present in the mean-field CF spectrum. Because so many results can be successfully interpreted in terms of CFs, the understanding of the actual reason for the success of the mean-field CF model, as well as defining its limitations and range of applicability, is extremely important.

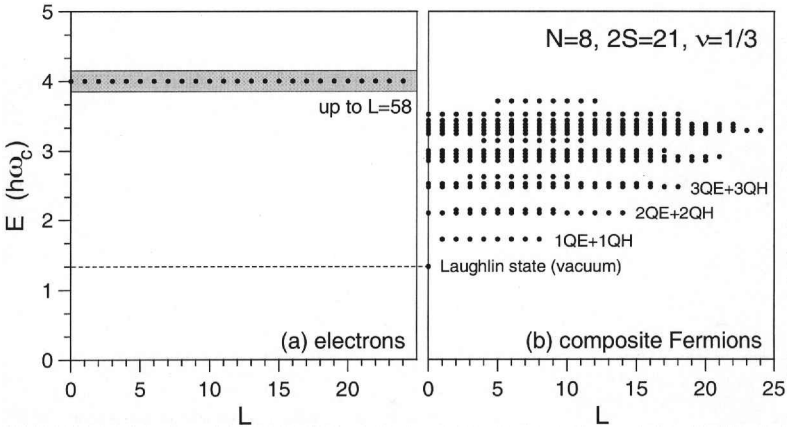

Figure 4. The energy spectra of (*a*) non-interacting electrons and (*b*) non-interacting CFs. The characteristic energy of the Coulomb interaction is indicated in (*a*) by a grey rectangle.

§ 4. Pseudopotential of the Coulomb interaction

The two-body interaction Hamiltonian of the many-body system can be expressed as

$$\hat{H} = \sum_{i<j} \sum_{L} V(L)\, \hat{\mathcal{P}}_{ij}(L). \tag{13}$$

Here, $V(L)$ is the two-particle interaction pseudopotential (Haldane 1987) defined as the interaction energy of a pair in the eigenstate $|L\rangle$ of angular momentum L,

$$\hat{H}|L\rangle = V(L)|L\rangle \tag{14}$$

and $\hat{\mathcal{P}}_{ij}(L)$ is the projection operator onto the subspace with the pair ij in the state $|L\rangle$. The pair angular momentum L measures the average squared electron–electron distance d^2. It can be shown that within the nth LL of the Haldane sphere

$$\frac{\hat{d}^2}{R^2} = 2 + \frac{S^2}{l(l+1)} \left(2 - \frac{\hat{L}^2}{l(l+1)} \right) \tag{15}$$

Note that $0 < d^2 < (2R)^2$ and $d^2 \equiv 2R^2$ for $2S = 0$.

Because of the confinement of single-particle states to one (lowest) LL, the number of pair states is strongly limited, and the electron–electron interaction potential enters the Hamiltonian H only through a small set of pseudopotential parameters. This reveals the magnetic field quantization of electron–electron interaction, that is electron–electron separation (Laughlin 1983b). On a Haldane sphere with a given $2S$, a finite number of these parameters, $V(2l - \mathcal{R})$, where $\mathcal{R} \leqslant 2l$ is an odd integer, determines many-body eigenstates and eigenenergies. Using the relative angular momentum $\mathcal{R}$ instead of the eigenvalue L of total angular momentum $\hat{\mathbf{L}} = \hat{\mathbf{l}}_1 + \hat{\mathbf{l}}_2$ to label pair states and pseudopotential coefficients allows meaningful comparison of the pseudopotentials in the planar system and in spherical systems with different l (or $2S$). On a sphere, $\mathcal{R}$ is defined as

$$\mathcal{R} = 2l - L, \tag{16}$$

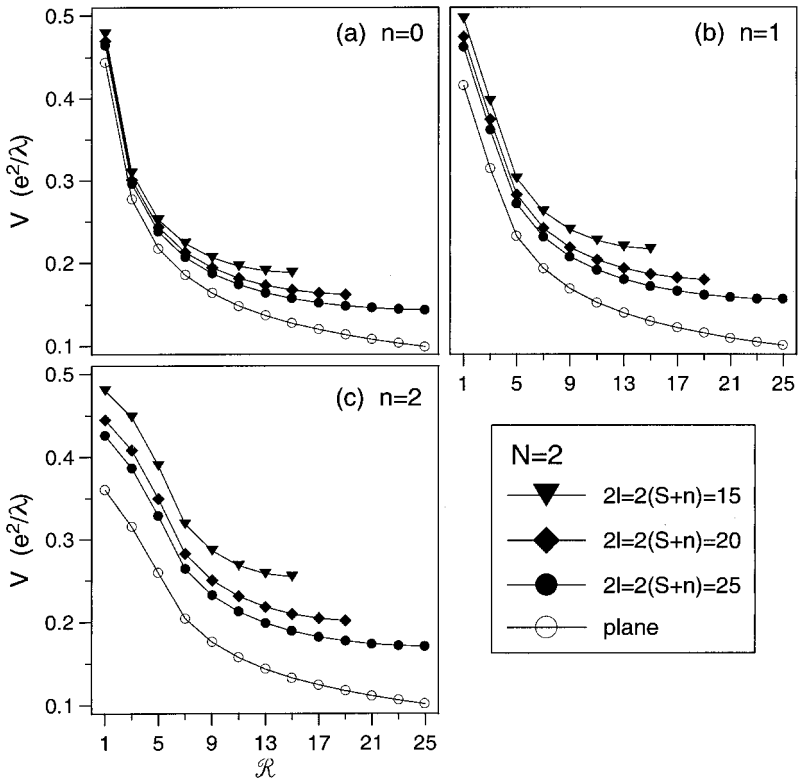

Figure 5. The pseudopotentials V of the Coulomb interaction in (a) the lowest, (b) the first excited and (c) the second excited LLs as functions of the relative angular momentum $\mathcal{R}$: (○), plane; (▼), Haldane sphere with $l = \frac{15}{2}$; (◆), Haldane sphere with $l = 10$; (●), Haldane sphere with $l = \frac{25}{2}$.

and on a plane it is equal to the angular momentum associated with the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. In both cases, larger $\mathcal{R}$ means larger separation (see equation (15) for the sphere). Figure 5 shows the pseudopotentials $V(\mathcal{R})$ calculated for the lowest and the first two excited LLs ($n = 0$, 1 and 2) for the plane and for the Haldane sphere with $l = \frac{15}{2}$, 10 and $\frac{25}{2}$. All pseudopotentials $V(\mathcal{R})$ in figure 5 decrease with increasing $\mathcal{R}$.

The important part of the pseudopotential spectrum is where its slope is the highest. It follows from equation (15) that each pair state with a given L corresponds to a certain average separation d and, roughly, $d \propto \mathcal{R}$. A large slope $\mathrm{d}V/\mathrm{d}\mathcal{R}$ means a large energy gradient, that is a large effective force, that would describe two point charges at a distance d. This effective force is solely due to the Coulomb force but takes into account different spreads of the electron wavefunctions in pair states for different $2S$, l and $\mathcal{R}$. As will be shown later, the crucial difference between the lowest LL (figure 5 (a)) and excited LLs (figures 5 (b) and (c)) is that, in the former case, $V(\mathcal{R})$ decreases more quickly at the smallest values of $\mathcal{R}$.

Let us define a model hard core pseudopotential V_{HC} for which

$$V_{\mathrm{HC}}(\mathcal{R}) \gg V_{\mathrm{HC}}(\mathcal{R}+2),$$
$$V_{\mathrm{HC}}(\mathcal{R}-2) - V_{\mathrm{HC}}(\mathcal{R}) \gg V_{\mathrm{HC}}(\mathcal{R}) - V_{\mathrm{HC}}(\mathcal{R}+2) \tag{17}$$

for all values of $\mathcal{R}$. The V_{HC} is an 'ideal' short-range pseudopotential (the class of short-range pseudopotentials leading to the similar Laughlin-like short-range correlations will be formally defined in § 7.5). The conditions (17) can be rewritten as $\mathrm{d}V/\mathrm{d}\mathcal{R} \ll 0$ and $\mathrm{d}^2V/\mathrm{d}\mathcal{R}^2 \gg 0$, where the derivatives are to be understood as finite differences. Clearly, in the low-lying many-body eigenstates of V_{HC}, electrons must avoid as much as possible pair states with largest repulsion, that is pair states with the smallest separation or smallest values of $\mathcal{R}$. The many-body states that avoid certain values of $\mathcal{R}$ can be constructed explicitly using parentage or grandparentage coefficients. In the following sections we shall investigate in detail the connection between the low-lying states of the FQH systems and the avoidance of pair states with largest repulsion.

§ 5. Three-electron system

5.1. *Coefficients of fractional parentage*

We begin the discussion of the three-electron case by listing in table 3 all possible L multiplets appearing in the spectrum for a given single-particle angular momentum l. An eigenfunction of three electrons each of angular momentum l whose total angular momentum is L will be denoted by $|l^3, L\alpha\rangle$, with an index α distinguishing different multiplets with the same L. This state can be written as

$$|l^3, L\alpha\rangle = \sum_{L_{12}} F_{L\alpha}(L_{12})|l^2, L_{12}; l, L\rangle, \tag{18}$$

a combination of product states $|l^2, L_{12}; l, L\rangle$ in which $l_1 = l_2 = l$ are added to obtain pair angular momentum L_{12}, and then $l_3 = l$ is added to L_{12} to obtain total angular

Table 3. The number of times that an L multiplet appears for a system of three electrons of angular momentum l. The top half of the table is for even values of $2l$, and the bottom half is for values of $2l$. Blank spaces are equivalent to zeros.

	Number of times that an L multiplet appears for the following $2L$ values																		
$2l$	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36
2	1																		
4		1		1															
6	1		1	1	1		1												
8		1		2	1	1	1	1		1									
10	1		1	1	2	1	2	1	1	1	1		1						
12		1		2		1	2	2	2	1	2	1	1	1	1		1		
14	1		1	1	2	1	3	2	2	2	2	1	2	1	1	1	1		1
$2l$	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37
3		1																	
5		1	1		1														
7		1	1	1	1	1		1											
9		1	1	1	2	1	1	1	1		1								
11		1	1	1	2	2	1	2	1	1	1	1		1					
13		1	1	1	2	2	2	2	2	1	2	1	1	1	1		1		

momentum L (de Shalit and Talmi 1963, Cowan 1981). Note that the state $|l^3, L\alpha\rangle$ is antisymmetric under interchange of any pair of particles 1, 2 and 3, while states $|l^2, L_{12}; l, L\rangle$ are antisymmetric only under interchange of particles 1 and 2. The factor $F_{L\alpha}(L_{12})$, or $F_{L\alpha}(\mathcal{R})$ where $\mathcal{R} = 2l - L_{12}$, is called the coefficient of fractional parentage (CFP) associated with pair angular momentum L_{12}.

The two-particle interaction matrix element can be conveniently expressed through the CFPs and the pseudopotential coefficients (Sitko *et al.* 1996):

$$\langle l^3, L\alpha | V | l^3, L\beta \rangle = 3 \sum_{\mathcal{R}} F_{L\alpha}(\mathcal{R}) F_{L\beta}(\mathcal{R})\, V(\mathcal{R}). \tag{19}$$

If the state $|l^3, L\alpha\rangle$ is an eigenstate of the interacting system, its energy is

$$E_{L\alpha} = 3 \sum_{\mathcal{R}} \mathcal{F}_{L\alpha}(\mathcal{R})\, V(\mathcal{R}), \tag{20}$$

where $\mathcal{F}_{L\alpha} = |F_{L\alpha}|^2$. The CFPs for three particles with given l can be derived analytically or found in nuclear (de Shalit and Talmi 1963) or atomic (Cowan 1981) physics books. Note, however, that the squared CFPs, which appear in equation (20) and measure the probability that a pair of electrons ij are in the pair state of angular momentum $\mathcal{R}$ can be expressed as

$$\mathcal{F}_{L\alpha}(\mathcal{R}) = \langle L\alpha | \hat{\mathcal{P}}_{ij}(\mathcal{R}) | L\alpha \rangle. \tag{21}$$

It follows from equation (13) that they can be calculated quite easily for any state $|L\alpha\rangle$ as the expectation value of the 'selective interaction' Hamiltonian $\hat{H}_{\mathcal{R}}$, whose only non-vanishing pseudopotential parameter is $V(\mathcal{R}) = 1$:

$$\mathcal{F}_{L\alpha}(\mathcal{R}) = \tfrac{1}{3} \langle L\alpha | \hat{H}_{\mathcal{R}} | L\alpha \rangle. \tag{22}$$

5.2. *Hard-core repulsive interaction*

For the hard-core pseudopotential defined in equation (17), the low-lying states must avoid low values of $\mathcal{R}$ as much as possible within the available Hilbert space. They have the maximum allowed number of vanishing CFPs which correspond to lowest values of $\mathcal{R}$, $\mathcal{F}_{L\alpha}(1) = \mathcal{F}_{L\alpha}(3) = \cdots = 0$. In such states, all pairs ij have zero projection on to pair states with a number of lowest values of $\mathcal{R}$,

$$\sum_{i<j} \sum_{\mathcal{R}=1,3,\ldots} \hat{\mathcal{P}}_{ij}(\mathcal{R}) | L\alpha \rangle = 0, \tag{23}$$

or with a number of pseudopotential parameters associated with the strongest repulsion, $V(1)$, $V(3), \ldots$.

For three electrons (fermions), the angular momenta of states in which $\mathcal{R} \geqslant 3, 5, \ldots$, for all pairs can be predicted from the following argument (Wójs and Quinn 1998b). If we choose $\mathcal{R} = 1$ for the pair of electrons 1 and 2 (i.e. $L_{12} = 2l - 1$) and add to L_{12} the same single-particle angular momentum l of the third electron, then the total angular momentum L must satisfy the vector addition rule, $|L_{12} - l| \leqslant L \leqslant L_{12} + l$. The antisymmetrization of the total wavefunction will eliminate some of the values of L from this range, but it is guaranteed that no states with L smaller than the minimum value, $L < l - 1$, can have non-vanishing parentage from $\mathcal{R} = 1$. In table 3, we have underlined the three-electron states with $L < l - 1$, which must therefore have $\mathcal{R} \geqslant 3$ for all pairs. The next higher value of $\mathcal{R}$ to avoid is 3, and, using the same argument as above, we obtain that all states with

Table 4. The allowed values of $2L$ for a three-electron system that must have $\mathcal{R} \geqslant 3$, 5 and 7. The listed values correspond to the underlined L multiplets in table 3.

	Allowed values for the following $2l$ values								
	6	7	8	9	10	11	12	13	14
$2L\ (\mathcal{R} \geqslant 3)$	0	3	2	3, 5	0, 4, 6	3, 5, 7	$2,6^2,8$	$3,5,7,9^2$	$0,4,6,8^2,10$
$2L\ (\mathcal{R} \geqslant 5)$					0	3	2	3, 5	0, 4, 6
$2L\ (\mathcal{R} \geqslant 7)$									0

$L < l-3$ must have $\mathcal{R} \geqslant 5$ (double underlined in table 3). Further, states with $L < l-5$ must all have $\mathcal{R} \geqslant 7$ (triple underlined in table 3), and so on. In table 4 we list the values of $2L$ for which the CFP with $\mathcal{R} = 1$ or with $\mathcal{R} \leqslant 3$ or with $\mathcal{R} \leqslant 5$ must vanish, that is $\mathcal{R} \geqslant 3$, 5 or 7 respectively. The $L = 0$ states for $2S = 6$, 10 and 14 are the Laughlin ground states with $\nu = \frac{1}{3}$, $\frac{1}{5}$, and $\frac{1}{7}$, respectively.

Note that the multiplets listed at $2l$ with $\mathcal{R} \geqslant \mathcal{R}^{\min}$ are always the same as those at $2l - 2p(N-1)$ with $\mathcal{R} \geqslant \mathcal{R}^{\min} - 2p$. However, for the lowest LL ($l = S$), $2S - 2p(N-1)$ is just $2S^*$, the effective monopole strength of CFs! This very important result remains true for any number of electrons, and will be discussed in more detail in § 6.2.

At $2S = 8$, two $L = 3$ multiplets occur (see table 3) and the interparticle interaction must be diagonalized in this two-dimensional subspace. The CFP for $\mathcal{R} = 1$ does not vanish identically in entire subspace because $L \geqslant l-1$. However, a linear combination can be constructed for which it does. For a model pseudopotential with $V(1) > 0$ and all other parameters vanishing, this would be the lower (zero-energy) eigenstate. At $2S = 14$ there are three allowed $L = 6$ multiplets, out of which one linear combination can be constructed with zero CFP for both $\mathcal{R} = 1$ and 3, and another without CFP for $\mathcal{R} = 1$ but with significant CFP for $\mathcal{R} = 3$.

5.3. *Coulomb interaction in the lowest and excited Landau levels*

How does this work for the actual Coulomb interaction? Figure 6 shows the Coulomb energy as a function of the total angular momentum L for the system of three electrons each with $l = 7$, that is at the filling factor $\nu = \frac{1}{7}$. Figure 6 (*a*) corresponds to the lowest LL and figures 6 (*b*) and (*c*) to two excited LLs. The insets show the spectra for $l = 3$ (filling factor $\nu = \frac{1}{3}$). Since the individual electron angular momentum $l = S + n$ is the same in figures 6 (*a*), (*b*) and (*c*), the three-electron Hilbert spaces contain the same L multiplets. The difference between the spectra in figures 6 (*a*), (*b*) and (*c*) comes from different Coulomb matrix elements, that is different pseudopotentials $V(\mathcal{R})$, in different LLs.

For the lowest LL, the Coulomb interaction plotted in figure 5 (*a*) behaves like the hard-core repulsion V_{HC} defined in equation (17). The energy spectrum in figure 6 (*a*) splits into bands of states with no parentage from pair states with $\mathcal{R} < 7$ (full diamonds), $\mathcal{R} < 5$ (open circles) and $\mathcal{R} < 3$ (full circles), and the remaining states with parentage from all pair states including $\mathcal{R} = 1$ (open squares). The CFPs, which are expected to vanish identically for any pseudopotential (see the last column in table 4) or which would vanish for the eigenstates of the interaction V_{HC} defined in equation (17), indeed vanish or are very small ($\mathcal{F} < 0.01$) for the eigenstates of the Coulomb interaction. This means that the Coulomb interaction within the lowest LL acts like V_{HC} and the two interactions have essentially identical eigenstates.

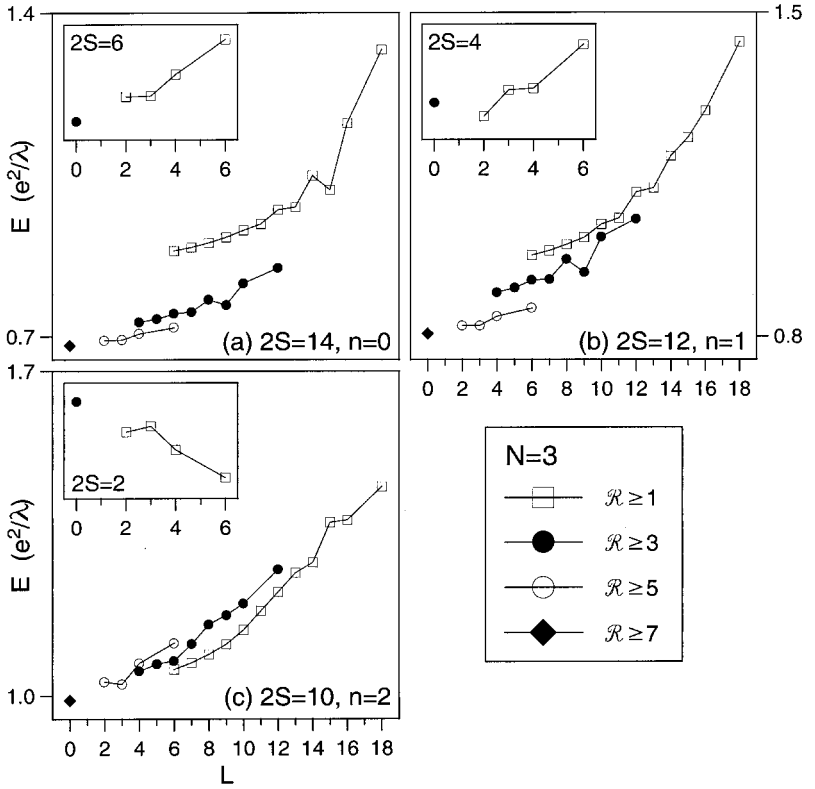

Figure 6. The Coulomb energy of three electrons each with $l = 7$ in (a) the lowest, (b) the first excited and (c) the second excited LLs: (◆) states with $R \geqslant 7$, that is $\mathcal{F}(1) \approx \mathcal{F}(3) \approx \mathcal{F}(5) \approx 0$ and $\mathcal{F}(7) > 0$; (○), $\mathcal{R} \geqslant 5$, that is $\mathcal{F}(1) \approx \mathcal{F}(3) \approx 0$ and $\mathcal{F}(5) > 0$; (●), $\mathcal{R} \geqslant 3$, that is $\mathcal{F}(1) \approx 0$ and $\mathcal{F}(3) > 0$; (□), $\mathcal{R} \geqslant 1$, that is $\mathcal{F}(1) > 0$. The ground states in all frames are the Laughlin $\nu = \frac{1}{7}$ states within different LLs. The insets show the spectra for $l = 3$; the ground state for $n = 0$ is the Laughlin $\nu = \frac{1}{3}$ state.

Since $V(1) - V(3) > V(3) - V(5) > \cdots$ in figure 5 (a), the gap between the highest-energy band ($\mathcal{R} \geqslant 1$) and the lower bands is the largest, the gap below the $\mathcal{R} \geqslant 3$ band is the next largest, etc. The lowest band ($\mathcal{R} \geqslant 7$) consists of only one state at $L = 0$. This is the Laughlin $\nu = \frac{1}{7}$ ground state. The excitation gap above the $\nu = \frac{1}{7}$ state is governed by $V(5) - V(7)$ and, as might be expected, it is almost unobservable. Note also that the first excited band in figure 6 (a) containing states with $\mathcal{R} \geqslant 5$ consists of multiplets at $L = 2$, 3, 4 and 6, in contrast with the mean-field CF prediction ($L = 1$, 2 and 3).

The inset in figure 6 (a) shows the spectrum for $l = 3$. The $L = 0$ ground state has $\mathcal{F}(1) = 0$ (see the first column in table 4); this is the Laughlin $\nu = \frac{1}{3}$ state. The structure of energy the spectrum for $l = 3$ is very similar to that within the two lowest bands for $l = 7$. This is because the Coulomb interaction for $n = 0$ acts like hard-core repulsion and decreasing angular momentum by $p(N-1)$ is equivalent to introduction of a hard core which forbids pair states with $\mathcal{R} < 2p + 1$ (see figure 7 and the discussion in the following section).

The Coulomb pseudopotentials for $n = 0$ in figure 5 (a) and for $n = 1$ in figure 5 (b) behave similarly for $\mathcal{R} \geqslant 3$. In consequence, the two lowest bands of states in

figure 6 (*a*) and (*b*) look similar. The CFPs, which are expected to be small, are found to be smaller than 0.01 for both $n = 0$ and $n = 1$. However, for the smallest $\mathcal{R}$, the condition $V(1) - V(3) \gg V(3) - V(5)$ is no longer satisfied for $n = 1$. Close to $\mathcal{R} = 1$, the Coulomb pseudopotential for $n = 1$ decreases too slowly with increasing $\mathcal{R}$, and its eigenstates, having some parentage from the $\mathcal{R} = 1$ pair state, are significantly different from those of the hard-core repulsion. For example, the states at $L = 10$ and 12 indicated by full circles in figure 6 (*b*) both have significant parentage from $\mathcal{R} = 1$, $\mathcal{F}(1) \approx 0.11$, while the two other states with $L = 10$ and 12, indicated by open squares, both have $\mathcal{F}(1) \approx 0.23$, only twice as large. For the same reason, there is almost no gap above the $\mathcal{R} \geqslant 3$ band for $n = 1$, in contrast with the $n = 0$ spectrum.

Different behaviour of $V(\mathcal{R})$ for $n = 1$ at small values of $\mathcal{R}$ has a much more pronounced effect on the $l = 3$ spectrum shown in the inset. The $L = 0$ state must have $\mathcal{F}(1) = 0$ because of the angular momentum addition argument (see table 4), but it is no longer the ground state. Let us stress this result: for three electrons, the Laughlin-like $\nu = \frac{1}{3}$ state is not the ground state in the first excited LL. Hence, the Laughlin-like $\nu = 2 + \frac{1}{3}$ state is not the ground state of the 13-electron system at $2S = 4$. However, the Laughlin-like $\nu = 2 + \frac{1}{7}$ state remains the ground state of 29 electrons at $2S = 12$.

For $n = 2$, the Coulomb pseudopotential in figure 5 (*c*) deviates from that for $n = 0$ at all $\mathcal{R} < 5$, and the only gap which persists in the spectrum in figure 6 is that above the $\mathcal{R} \geqslant 7$ ground state. Higher bands, containing states with the smallest appropriate CFP (which would be zero for the hard-core repulsion) are not even ordered as those for $n = 0$ or 1. In the inset, the Laughlin $\nu = \frac{1}{3}$ state with $\mathcal{R} \geqslant 3$ is the highest-energy state for $n = 2$.

§ 6. Many-electron systems

6.1. *Coefficients of fractional grandparentage*

Equations (18) and (19) can be generalized to the case of an arbitrary number of electrons. An antisymmetric wavefunction $|l^N, L\alpha\rangle$ of N electrons each with angular momentum l that are combined to give a total angular momentum L can be written as (de Shalit and Talmi 1963, Cowan 1981)

$$|l^N, L\alpha\rangle = \sum_{L_{12}} \sum_{L'\alpha'} G_{L\alpha, L'\alpha'}(L_{12}) |l^2, L_{12}; l^{N-2}, L'\alpha'; L\rangle. \tag{24}$$

Here, $|l^2, L_{12}; l^{N-2}, L'\alpha'; L\rangle$ denote product states in which angular momenta $l_1 = l_2 = l$ of two electrons are added to obtain the pair angular momentum L_{12}, then angular momenta $l_3 = l_4 = \cdots = l_N = l$ of remaining $N - 2$ electrons are added to obtain the angular momentum L' (different states with this angular momentum are labelled with different α'), and finally L_{12} is added to L' to obtain the total angular momentum L. The state $|l^N, L\alpha\rangle$ is totally antisymmetric, while the states $|l^2, L_{12}; l^{N-2}, L'\alpha'; L\rangle$ are antisymmetric under interchange of particles 1 and 2, and under interchange of any pair of particles $3, 4, \ldots, N$. The factor $G_{L\alpha, L'\alpha'}(L_{12})$, or $G_{L\alpha, L'\alpha'}(\mathcal{R})$ where $\mathcal{R} = 2l - L_{12}$, is called the coefficient of fractional grandparentage (CFGP). For $N = 3$, it is equivalent to the CFP: $G_{L\alpha, l}(\mathcal{R}) \equiv F_{L\alpha}(\mathcal{R})$.

The two-particle interaction matrix element expressed through the CFGPs is

$$\langle l^N, L\alpha | V | l^N, L\beta \rangle = \frac{N(N-1)}{2} \sum_{\mathcal{R}} \sum_{L'\alpha'} G_{L\alpha,L'\alpha'}(\mathcal{R})\, G_{L\beta,L'\alpha'}(\mathcal{R})\, V(\mathcal{R}). \tag{25}$$

For an interaction eigenstate, its energy is

$$E_{L\alpha} = \frac{N(N-1)}{2} \sum_{\mathcal{R}} \mathcal{G}_{L\alpha}(\mathcal{R})\, V(\mathcal{R}), \tag{26}$$

where the coefficient

$$\mathcal{G}_{L\alpha}(\mathcal{R}) = \sum_{L'\alpha'} |G_{L\alpha,L'\alpha'}(\mathcal{R})|^2 \tag{27}$$

gives the probability that a pair of electrons ij are in the pair state of a given $\mathcal{R}$. The derivation of the CFGPs for arbitrary N and l is rather tedious. Note, however, that the coefficients $\mathcal{G}(\mathcal{R})$ can be expressed as (compare equation (21))

$$|\mathcal{G}_{L\alpha}(\mathcal{R})|^2 = \langle L\alpha | \hat{\mathcal{P}}_{ij}(\mathcal{R}) | L\alpha \rangle \tag{28}$$

and calculated as the expectation value of the 'selective interaction' Hamiltonian $\hat{H}_{\mathcal{R}}$, whose only non-vanishing pseudopotential parameter is $V(\mathcal{R}) = 1$ (compare equation (22)):

$$\mathcal{G}_{L\alpha}(\mathcal{R}) = \frac{2}{N(N-1)} \langle L\alpha | \hat{H}_{\mathcal{R}} | L\alpha \rangle. \tag{29}$$

From the orthonormality of functions $|l^N, L\alpha\rangle$ it is also apparent that

$$\sum_{\mathcal{R}} \sum_{L'\alpha'} G_{L\alpha,L'\alpha'}(\mathcal{R})\, G_{L\beta,L'\alpha'}(\mathcal{R}) = \delta_{\alpha\beta}. \tag{30}$$

6.2. *Dynamical symmetry of hard-core repulsion*

The angular momentum addition argument fails for more than three electrons, and there are no L multiplets for $N > 3$ whose CFGP for $\mathcal{R} = 1, 3, \ldots$ would vanish regardless of the form of interaction pseudopotential. However, the many-electron Hilbert space $\mathcal{H}$ still contains subspaces $\mathcal{H}_p$ holding many-body states with grandparentage only from pair states with $\mathcal{R} \geqslant 2p+1$, for which $\mathcal{G}(1) = \mathcal{G}(3) = \cdots = \mathcal{G}(2p-1) = 0$:

$$\mathcal{H} \equiv \mathcal{H}_0 \supset \mathcal{H}_1 \supset \mathcal{H}_2 \supset \cdots. \tag{31}$$

The total Hilbert space splits thus into subspaces $\tilde{\mathcal{H}}_p = \mathcal{H}_p \backslash \mathcal{H}_{p+1}$, containing many-body states that do not have grandparentage from pair states with $\mathcal{R} < 2p+1$ but have some grandparentage from $\mathcal{R} = 2p+1$:

$$\mathcal{H} = \tilde{\mathcal{H}}_0 \oplus \tilde{\mathcal{H}}_1 \oplus \tilde{\mathcal{H}}_2 \oplus \cdots. \tag{32}$$

For N electrons on a Haldane sphere each with angular momentum l, there is more than one subspace (subspace $\tilde{\mathcal{H}}_1$ is not empty) for $2l \geqslant 3(N-1)$, that is for filling factors $\nu \leqslant \frac{1}{3}$. In general, $\tilde{\mathcal{H}}_p$ is not empty (some states with $\mathcal{R} \geqslant 2p+1$ can be constructed) for $\nu \leqslant (2p+1)^{-1}$.

The subspaces $\tilde{\mathcal{H}}_p$ are the eigensubspaces of the hard-core repulsive potential V_{HC} defined in equation (17), whose low-energy states have to avoid grandparentage from pair states with large repulsion (small $\mathcal{R}$). Consequently, as for three electrons, the energy levels in the many electron spectrum with hard-core interaction form

bands corresponding to subspaces $\tilde{\mathcal{H}}_p$. For given N and l, that is for a given filling factor ν such that $(2p+3)^{-1} < \nu \leqslant (2p+1)^{-1}$, there are $p+1$ bands, and the qth band ($q = 0,1,\ldots,p$) corresponds to $\tilde{\mathcal{H}}_q$. The pth band is the lowest-energy band with the maximum number of CFGPs vanishing, and the zeroth band is the highest-energy band containing states with some grandparentage from the $\mathcal{R} = 1$ pair state. The energy gap between the qth band and the $(q+1)$th band is of the order of $V(2q+1) - V(2q+3)$. Hence, the largest gap is that between the zeroth band and the first band, the next largest is that between the first band and second band, etc.

Importantly, the set of angular momentum multiplets which make the qth band ($\tilde{\mathcal{H}}_q$ subspace) of the spectrum of N electrons each with angular momentum l is always the same as the set of multiplets in the $(q+1)$th band ($\tilde{\mathcal{H}}_{q+1}$ subspace) of N electrons each with angular momentum $l + (N-1)$. This is demonstrated in figure 7 for four electrons in the lowest LL interacting through the (hard-core-like) Coulomb pseudopotential. When $l = S$ is increased by $N-1$, the only significant difference in the spectrum is the appearance of an additional band at high energy. The structure of the low-energy part of the spectrum is completely unchanged. All

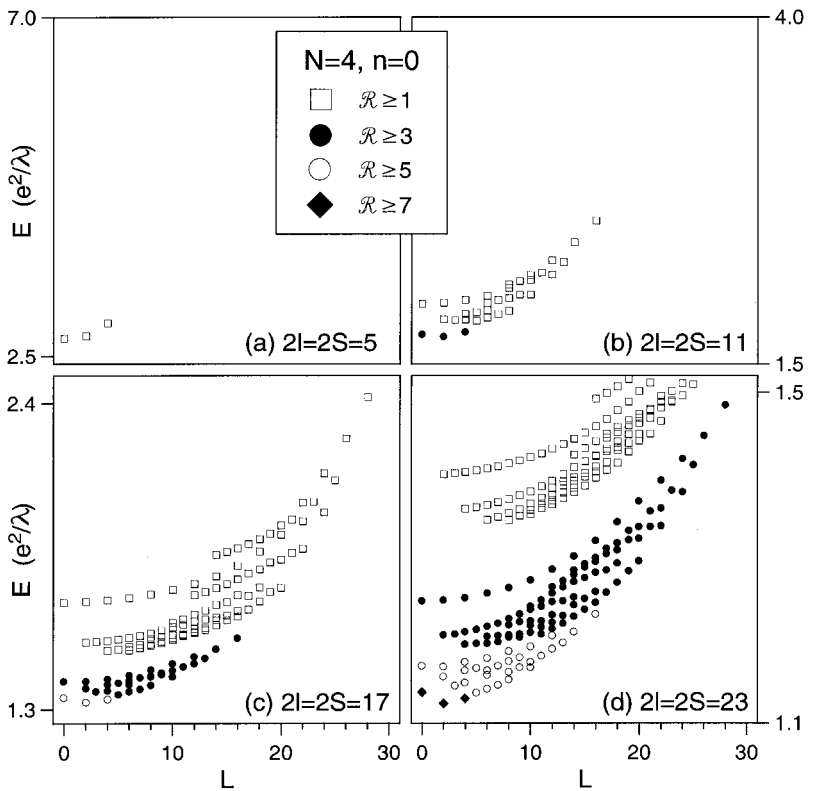

Figure 7. The energy spectra of four electrons in the lowest Landau level at different monopole strengths of (*a*) $2S = 5$; (*b*) $2S = 11$; (*c*) $2S = 17$; and (*d*) $2S = 23$ (all those $2S$ values are equivalent in the mean-field CF picture (CS transformation with $p = 0$, 1, 2 and 3 respectively)): (◆) states with $\mathcal{R} \geqslant 7$, that is $\mathcal{G}(1) \approx \mathcal{G}(3) \approx \mathcal{G}(5) \approx 0$ and $\mathcal{G}(7) > 0$; (○), $\mathcal{R} \geqslant 5$, that is $\mathcal{G}(1) \approx \mathcal{G}(3) \approx 0$ and $\mathcal{G}(5) > 0$; (●), $\mathcal{R} \geqslant 3$, that is $\mathcal{G}(1) \approx 0$ and $\mathcal{G}(3) > 0$; (□), $\mathcal{R} \geqslant 1$, that is $\mathcal{G}(1) > 0$.

bands and multiplets in the spectrum for $2S$ correspond directly to appropriate bands and multiplets in the spectrum for the monopole strength $2S+2(N-1)$. For example, all three allowed multiplets at $2S=5$ ($L=0$, 2, and 4) form the low-energy band at $2S=11$, 17 and 23, where they span the $\tilde{\mathcal{H}}_1$, $\tilde{\mathcal{H}}_2$ and $\tilde{\mathcal{H}}_3$ subspaces respectively. Similarly, the first excited band at $2S=11$ (open squares in figure 7 (*b*)) is repeated in the spectra for $2S=17$ and 23, where it corresponds to $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_2$ subspaces, respectively.

Let us repeat that the fact that identical multiplets occur in subspace $\tilde{\mathcal{H}}_q$ for the single-electron angular momentum l, and in subspace $\tilde{\mathcal{H}}_{q+1}$ for l replaced by $l+(N-1)$, does not depend on the form of interaction and follows solely from the rules of addition of angular momenta of identical fermions. However, if the interaction pseudopotential has the hard-core properties as in equation (17), then the many-body interaction Hamiltonian has a new dynamical symmetry, as a result of which the following hold.

(i) The subspaces $\tilde{\mathcal{H}}_q$ are the eigensubspaces and the subspace (band) index q is a good quantum number.
(ii) The energy bands corresponding to $\tilde{\mathcal{H}}_q$ with higher q lie below those of lower q.
(iii) The spacing between neighbouring bands is governed by a difference between appropriate pseudopotential coefficients.
(iv) The wavefunctions and the relative position of energy levels within each (qth) band do not depend on the details of interaction (it will be shown later that they repeat the spectrum of $\mathcal{G}(2q+1)$; see figure 15).

Replacing the model hard-core pseudopotential by a 'softer' pseudopotential (the measure of the 'hard core' character β will be specified in § 7.5 leads to, firstly, coupling between subspaces $\tilde{\mathcal{H}}_q$, secondly, mixing, overlap or even order reversal of bands and, thirdly, deviation of wavefunctions and the structure of energy levels within bands from those of the hard-core repulsion (and thus their dependence on details of the interaction pseudopotential).

The reoccurrence of L multiplets forming the low-energy band when l is replaced by $l\pm(N-1)$ has the following crucial implication. The lowest-energy pth band contains L multiplets which are all the allowed multiplets of N electrons each with angular momentum $l-p(N-1)$. This is because, if $(2p+3)^{-1}<\nu_{N,l}\leqslant(2p+1)^{-1}$, then $\frac{1}{3}<\nu_{N,l-p(N-1)}\leqslant 1$ and there is only one (zeroth) band in the spectrum. As for three electrons, for the lowest LL with $l=S$ this means that the lowest-energy band at the monopole strength $2S$ contains a subset of low-energy multiplets which are all the allowed multiplets at a smaller monopole strength $2S-2p(N-1)$, but $2S-2p(N-1)$ is just $2S^*$, the effective monopole strength of CFs! The mean-field CS transformation which attaches $2p$ fluxes (vortices) to each electron selects the same L multiplets from the entire spectrum as does the introduction of a hard core, which forbids a pair of electrons to be in a state with $\mathcal{R}<2p+1$.

The success of the mean-field CF picture in the prediction of the low-lying band of states in the many-electron spectrum relies on the fact that the Coulomb interaction within the lowest LL acts like the hard-core repulsion. For filling factors ν such that $(2p+3)^{-1}<\nu\leqslant(2p+1)^{-1}$, the states predicted by the mean-field CF picture as the states of an appropriate number of QHs in the Laughlin $\nu=(2p+1)^{-1}$ ground state are the states which for the hard-core interaction have the maximum number p of vanishing CFGPs associated with the highest pseudopotential para-

meters. These are the states with $\mathcal{R} \geqslant 2p+1$ spanning the subspace $\tilde{\mathcal{H}}_p$. In particular, there is always only one state with $\mathcal{R} \geqslant 2p+1$ ($\tilde{\mathcal{H}}_p$ is one dimensional) at the filling factor $\nu = (2p+1)^{-1}$. This state has $L = 0$ and it is the Laughlin incompressible ground state, separated from other states by the gap Δ which is of the order of $\Delta = V(2p-1) - V(2p+1)$.

As long as the eigenstates of the Coulomb interaction are approximately those of the hard-core repulsive interaction, the incompressible ground states are associated with the appearance of states with significantly lower CFGPs than all other states in the spectrum. The Laughlin $\nu = (2p+1)^{-1}$ ground states are the only states with $\mathcal{G}(1) \approx \mathcal{G}(3) \approx \cdots \approx \mathcal{G}(2p-1) \approx 0$ in their Hilbert spaces (the CFGPs do not vanish identically because of the weak mixing between $\tilde{\mathcal{H}}_q$ subspaces). All other states have some (significant) grandparentage from pair states with $\mathcal{R} < 2p+1$. The Jain states at filling factors ν in the range $(2p+3)^{-1} < \nu < (2p+1)^{-1}$ are those of all states with $\mathcal{G}(1) \approx \mathcal{G}(3) \approx \cdots \approx \mathcal{G}(2p-1) \approx 0$, for which $\mathcal{G}(2p+1)$, the first non-vanishing CFGP, is significantly smaller than for other states (Wójs and Quinn 1999a).

What is the condition for the interaction pseudopotential to behave like the hard-core repulsion and have the energy spectrum characteristic of the FQH effect? In the following sections we answer this question and explain why the hard-core type (FQH) ground states occur for the Coulomb interaction within the lowest LL. We also show that, owing to a different form of the Coulomb pseudopotential in higher (spin-polarized) LLs, the FQH ground states for $n > 0$ occur only at lower densities, when, at low energy, only the hard-core-like part of the pseudopotential (at high $\mathcal{R}$) contributes to the Hamiltonian given by equation (13).

6.3. *Coulomb interaction in the lowest and excited Landau levels*

Figure 8 shows the Coulomb energy as a function of L for the system of four electrons each with $l = \frac{15}{2}$. Figure 8 (*a*) corresponds to the lowest LL ($n = 0$) and figures 8 (*b*) and (*c*) to the two excited LLs ($n = 1$ and 2); the insets show the spectra for $l = \frac{9}{2}$. Figure 8 is very similar to figure 6 and demonstrates that the conclusions drawn for the simple three-electron system remain valid for an arbitrary N.

As for three electrons, the Coulomb interaction within the lowest LL ($n = 0$) behaves like the hard-core interaction and the energy spectrum splits into bands of states with $\mathcal{R} \geqslant 5$ (open circles), $\mathcal{R} \geqslant 3$ (full circles) and $\mathcal{R} \geqslant 1$ (open squares). The $\mathcal{R} \geqslant 5$ band contains only the Laughlin $\nu = \frac{1}{5}$ ground state. For $N > 3$, none of CFPs vanishes identically for an arbitrary interaction, but the CFPs that would vanish for the eigenstates of the hard-core interaction defined in equation (17) indeed vanish or are very small ($\mathcal{G} < 0.01$) for the eigenstates of the Coulomb interaction. The inset in figure 8 (*a*) shows the spectrum for $l = \frac{9}{2}$, with the Laughlin $\nu = \frac{1}{3}$ ground state. The energy spectrum for $l = \frac{9}{2}$ repeats main features of the two lowest-energy bands for $l = \frac{15}{2}$.

Within the first excited LL ($n = 1$), only the lowest band with $\mathcal{R} \geqslant 5$ can be distinguished. The two higher bands ($\mathcal{R} \geqslant 3$ and $\mathcal{R} \geqslant 1$) overlap. Also, some of the coefficients $\mathcal{G}(1)$, which would be zero for the hard-core repulsion, are quite large (> 0.1) for $n = 1$. In the inset, the two $L = 0$ states have $\mathcal{G}(1) = 0.08$ and 0.26, and the Laughlin-like $\nu = \frac{1}{3}$ state with a smaller $\mathcal{G}(1)$ (full circle) is the state with a higher energy. Even though the ground state has $L = 0$, it is not the state with the Laughlin-like correlations, with electrons avoiding pair states with the largest repulsion (i.e. smallest average separation; see equation (15)). The gap above this

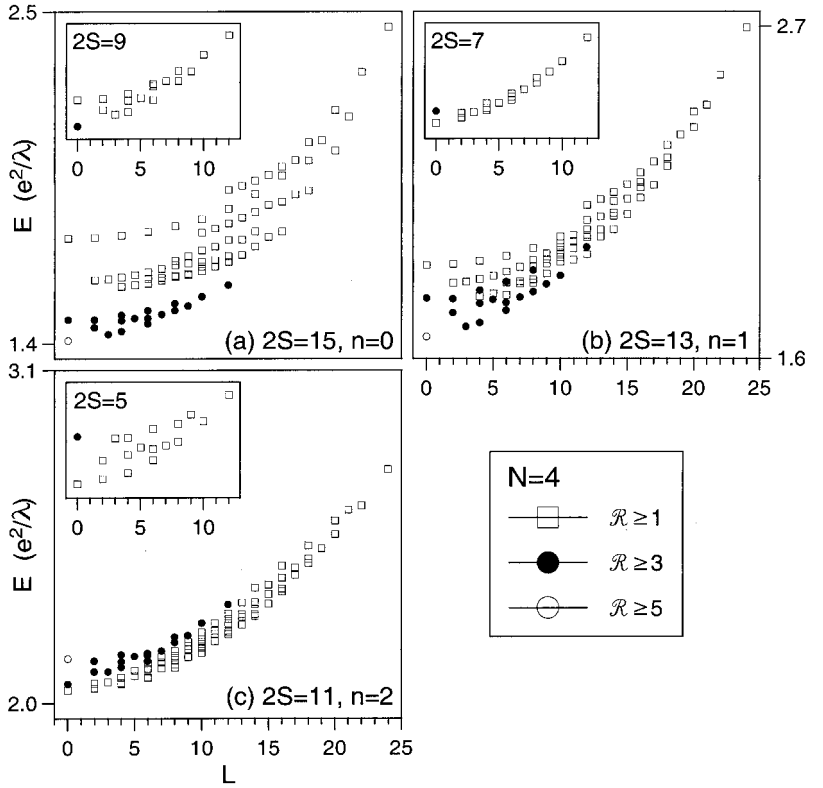

Figure 8. The Coulomb energy of four electrons each with $l = \frac{15}{2}$ in (*a*) the lowest, (*b*) the first excited and (*c*) the second excited LLs: (○) states with $\mathcal{R} \geqslant 5$, that is $\mathcal{G}(1) \approx \mathcal{G}(3) \approx 0$ and $\mathcal{G}(5) > 0$; (●), $\mathcal{R} \geqslant 3$, that is $\mathcal{G}(1) \approx 0$ and $\mathcal{G}(3) > 0$; and (□), $\mathcal{R} \geqslant 1$, that is $\mathcal{G}(1) > 0$. The ground states for $n = 0$ and 1 are the Laughlin $\nu = \frac{1}{5}$ states within these LLs. The insets show the spectra for $l = \frac{9}{2}$; the ground state for $n = 0$ is the Laughlin $\nu = \frac{1}{3}$ state.

ground state is not associated with the energy $V(1) - V(3)$, and hence the $\nu = 2 + \frac{1}{3}$ state is unlikely to be an incompressible ground state in the thermodynamic limit.

For $n = 2$, neither the Laughlin-like $\nu = \frac{1}{5}$ state in figure 8 (*c*) ($\mathcal{R} \geqslant 5$, open circle), nor the Laughlin-like $\nu = \frac{1}{3}$ state in the inset ($\mathcal{R} \geqslant 3$, full circle) is the ground state. This suggests that neither the $\nu = 4 + \frac{1}{3}$ state nor the $\nu = 4 + \frac{1}{5}$ state is an incompressible ground state in the thermodynamic limit.

We have calculated the energy spectra analogous to those in figure 8 for different numbers of electrons and conclude that the Laughlin-like $L = 0$ state with $\nu = \frac{1}{3}$, which is the only state with $\mathcal{R} \geqslant 3$ in its spectrum, is the ground state only within the lowest LL ($n = 0$). Similarly, the Laughlin-like $\nu = \frac{1}{5}$ state with $\mathcal{R} \geqslant 5$ is the ground state only for $n \leqslant 1$.

The angular momentum L of the ground state of N electrons at the monopole strength $2S$ corresponding to the $\nu = \frac{1}{3}$ filling within the LL of $n > 0$ or to the $\nu = \frac{1}{5}$ filling within the LL of $n > 1$ depends on N. Even though $L = 0$ (ground state is non-degenerate) for some values of N, the low-lying spectra do not resemble those in the lowest LL, and the excitation is not associated with energy $V(1) - V(3)$. In order to verify whether the $L = 0$ ground states with $\nu = 2 + \frac{1}{3}$, $2 + \frac{1}{5}$, $4 + \frac{1}{3}$, and $4 + \frac{1}{5}$ are

incompressible ground states in the thermodynamic limit, we have calculated the energy gaps above these states for different values of N. The energy spectra of up to 11 electrons at the filling factor $\nu = \frac{1}{3}$ in the lowest and first excited LLs are presented in figure 9. The energy scales for $n = 0$ and 1 are different, and the bar in the bottom right corner of each $n = 1$ graph on the right shows the energy gap of the corresponding system in the lowest LL on the left. Figure 10 shows the dependence of the gap $\Delta_{L=0}$ from the lowest $L = 0$ state to the lowest state of $L > 0$, as a function of N^{-1}. For filling factors $\nu = 2n + \frac{1}{3}$ (full circles), N varies between four and 11 and, for $\nu = 2n + \frac{1}{5}$ (open circles), N goes up to eight. Negative $\Delta_{L=0}$ means that the ground state is degenerate (has $L > 0$). In this case, $|\Delta_{L=0}|$ gives the excitation energy from this degenerate ground state to the lowest state at $L = 0$.

For $n = 0$, the ground states at both $\nu = \frac{1}{3}$ and $\frac{1}{5}$ are Laughlin incompressible states. The gap Δ persists for $N \to \infty$, and the estimates obtained from the best linear fits (broken lines) are $\Delta_{\nu=1/3} = 0.0632\, e^2/\lambda$ and $\Delta_{\nu=1/5} = 0.0105\, e^2/\lambda$. For $n = 1$, the $L = 0$ state at $\nu = 2 + \frac{1}{5}$ is the Laughlin-like state and the gap above it seems to converge to a finite value; the linear fit gives $\Delta_{\nu=2+1/5} = 0.0116\, e^2/\lambda$, very close to $\Delta_{\nu=1/5}$. On the other hand, the dependence of the gap Δ above the (non-Laughlin-like) states at $\nu = 2 + \frac{1}{3}$, $4 + \frac{1}{3}$, and $4 + \frac{1}{5}$ on the electron number N is quite different from that for Laughlin states. No conclusive statement about the incompressibility (or even the sign of Δ, that is the non-degeneracy) of these states in the thermodynamic limit can be made on the basis of our finite-size calculations for up to 11 electrons. Since at least at $\nu = 2 + \frac{1}{3}$ the FQH plateau has been observed experimentally (Willet *et al.* 1987), we have to restrict ourselves to repeating the statement (MacDonald and Girvin 1986) that the nature of the low-lying states at $\nu = 2 + \frac{1}{3}$, $4 + \frac{1}{3}$, and $4 + \frac{1}{5}$ is different from that of the Laughlin $\nu = \frac{1}{3}$ and $\frac{1}{5}$ states. In general, low-lying states in the lowest and nth LLs have Lauglin-like correlations only below the filling factor $\nu = (2n+1)^{-1}$. At fillings $\nu \geqslant (2n+1)^{-1}$ in the nth LL, the correlations are different, possible incompressibility has a different origin, the excitation gap is not simply related to the difference between appropriate pseudopotential parameters, and the excitations do not contain Laughlin QPs.

A clear signature of the non-Laughlin-like character of the $n = \frac{1}{3}$ state in excited LLs is the lack of QP-type excitations at neighbouring filling factors. In figure 11 we compare the energy spectra of ten electrons at equal fillings (near $\nu = \frac{1}{3}$) of the lowest and first excited LLs. In the lowest LL, lowest-energy states (indicated by solid lines) contain two QEs (figure 2 (*a*)), one QE (figure 2 (*c*)), one QH (figure 2 (*e*)) and two QHs (figure 2 (*g*)) in the Laughlin $\nu = \frac{1}{3}$ state, while in the first excited LL no similar low-lying states occur (note also that the energy axes for $n = 1$ are stretched by a factor of two compared with those for $n = 0$). Note also that the energies connected with the solid lines in figures 11 (*a*) and (*g*) define the pseudopotentials of a pair of appropriate interacting QPs in the Laughlin $\nu = \frac{1}{3}$ state.

For a complete report of our numerical results for the lowest LL, let us add a few values to the tables published earlier (Fano *et al.* 1986). In table 5 we list the Laughlin ground-state energy per particle (calculated including the interaction with a charge-compensating background, $-N^2e^2/2R$), the angular momentum and excitation energy of the magnetoroton minimum, and the 'proper' QE and QH energies (calculated including additional fractional charge $\pm e/m$ in the background (Haldane and Rezayi 1985a, Fano *et al.* 1986)), for $N = 10$ and 11 electrons at filling factor $\nu = \frac{1}{3}$ and for $N = 7$ and 8 electrons at $\nu = \frac{1}{5}$. The limiting values for $N \to \infty$ have been calculated using data for these and smaller values of N. For example, the

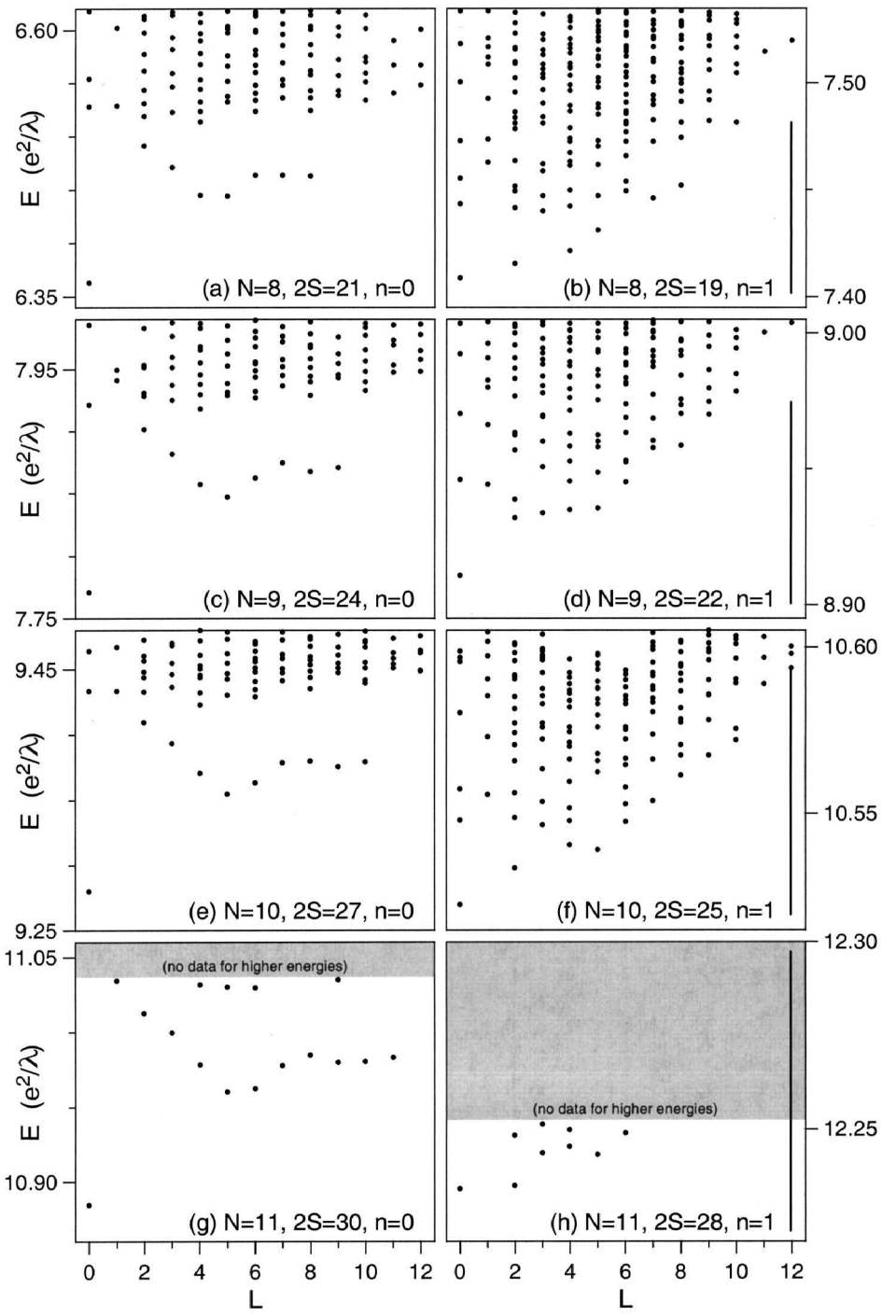

Figure 9. The energy spectra of (*a*), (*b*) eight, (*c*), (*d*) nine, (*e*), (*f*) ten and (*g*), (*h*) 11 electrons in (*a*), (*c*), (*e*), (*g*) the lowest and (*b*), (*d*), (*f*), (*h*) the first excited LLs at the filling factor $\nu = \frac{1}{3}$.

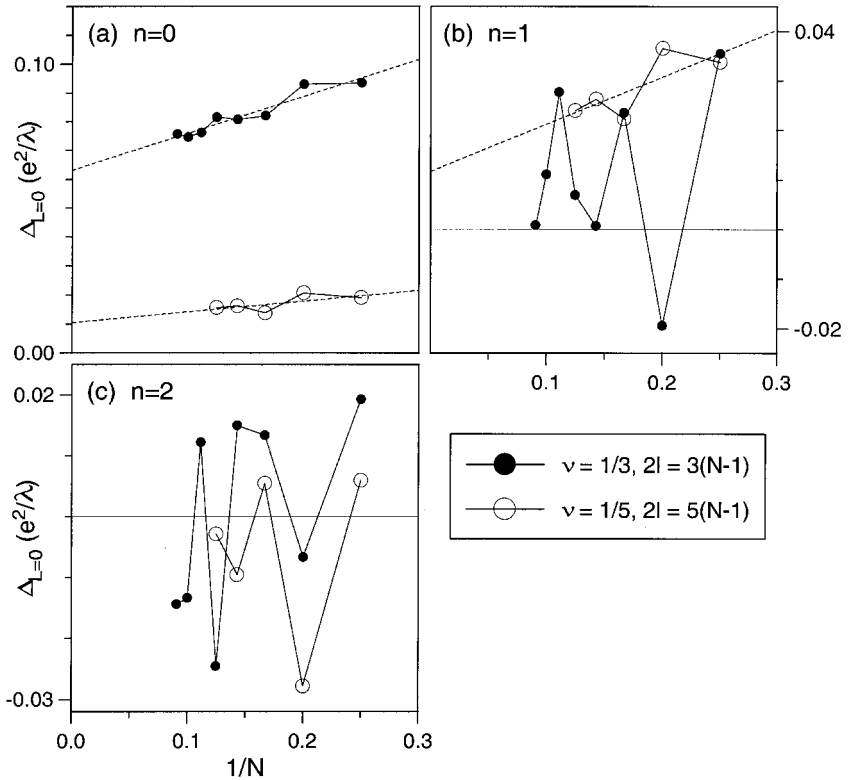

Figure 10. The energy gap $\Delta_{L=0}$ from the lowest $L = 0$ state to the lowest state at $L > 0$ as a function of the inverse electron number N^{-1}, for (*a*) the lowest, (*b*) the first excited and (*c*) the second excited LLs: (●), $\nu = 2n + \frac{1}{3}$; (○), $\nu = 2n + \frac{1}{5}$. (- - - -), linear fits for the Laughlin-like incompressible ground states at $\nu = \frac{1}{3}$, $\nu = \frac{1}{5}$, and $\nu = 2 + \frac{1}{5}$. The ground states at $\nu = 2 + \frac{1}{3}$, $4 + \frac{1}{3}$ and $4 + \frac{1}{5}$ are unlikely to be incompressible for $N \to \infty$.

QE and QH energies agree very well with extrapolation of the Monte Carlo results in disc geometry: $\varepsilon_{\mathrm{QE}} = 0.073$ and $\varepsilon_{\mathrm{QH}} = 0.0268$ (Morf and Halperin 1986).

It is known (Haldane and Rezayi 1985a) that the QE–QH excitonic energy dispersion (QE–QH pseudopotential) in a Laughlin state, calculated for a finite N-electron system and plotted as a function of the wave-vector $k = L/R$, quickly converges to the continuous curve of an infinite 2DEG, with a magnetoroton minimum at k of the order of the inverse magnetic length λ^{-1}. In figure 12 we present the QE–QH dispersion for the $\nu = \frac{1}{3}$ state, including data for up to 11 electrons. The continuum indicated by the grey rectangle starts at the lowest excitation energy of 11 electrons above the magnetoroton curve. The $\varepsilon_{\mathrm{QE}} + \varepsilon_{\mathrm{QH}} = 0.099\,492$ energy (our thermodynamic limit estimate) gives the energy of a QE–QP pair at an infinite distance (infinite k). The smooth solid curve connects the data points for $N = 11$.

6.4. *Grandparentage coefficients of low-lying states*

Typical dependences of $\mathcal{G}_{L\alpha}$ on $\mathcal{R}$ for low-lying states are plotted in figure 13 for a six-electron system at $l = \frac{11}{2}$ ($\nu = \frac{2}{5}$) and $l = \frac{15}{2}$ ($\nu = \frac{1}{3}$), in the lowest and first two excited LLs. The thicker lines and full circles indicate data corresponding to the state at $L = 0$. The CFGP profile $\mathcal{G}(\mathcal{R})$ can be regarded as a pair correlation function,

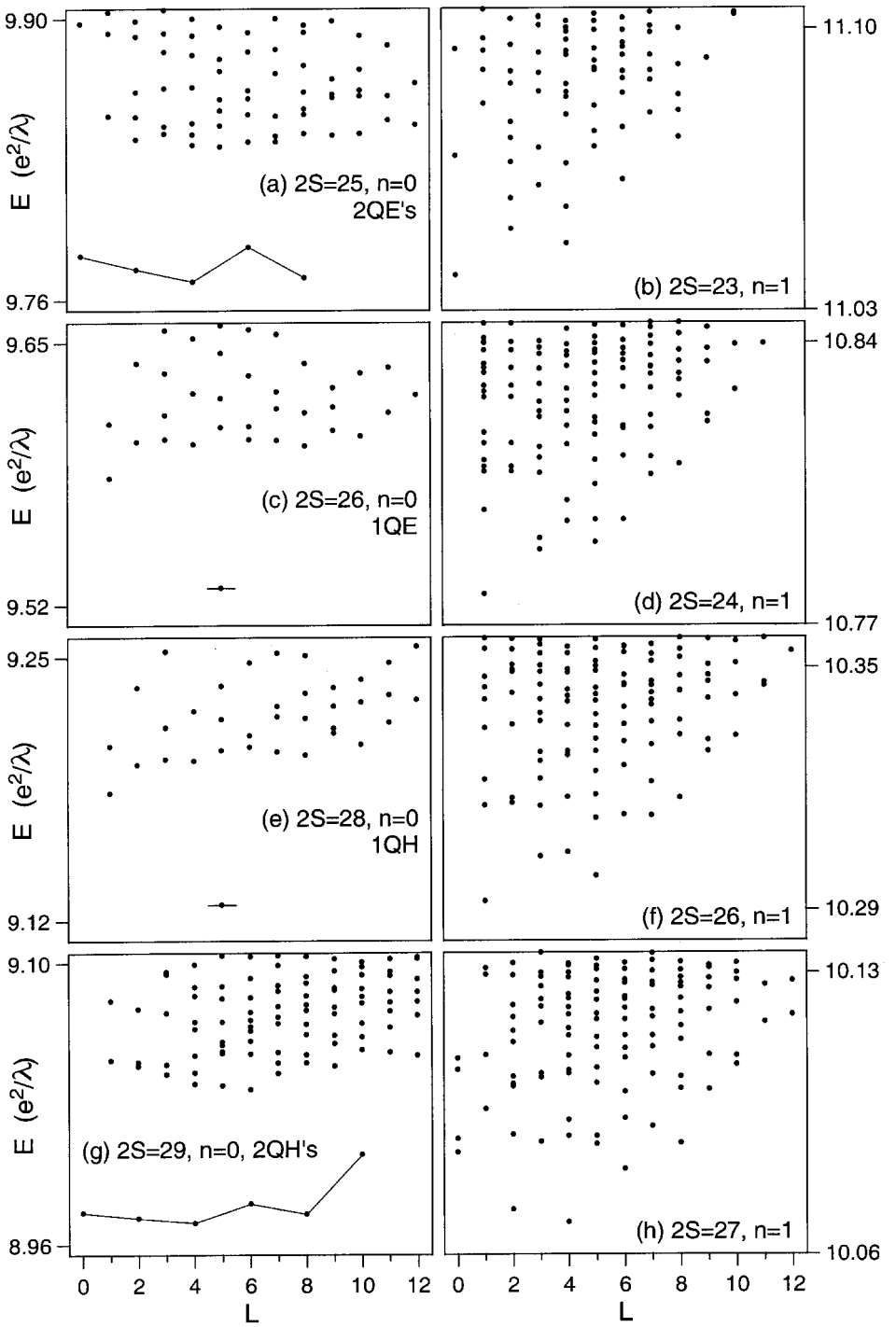

Figure 11. The energy spectra of ten electrons in (*a*), (*c*), (*e*), (*g*) the lowest and (*b*), (*d*), (*f*), (*h*) first excited LLs at filling factors near $\nu = \frac{1}{3}$.

Table 5. The ground energy per particle E/N of the Laughlin ground state, the angular momentum L and excitation energy Δ of the magnetoroton minimum, the proper QE energy, ϵ_{QE} and the proper QH energy ϵ_{QH} for N electrons at a filling factor ν.

ν	N	E/N	L	Δ	ϵ_{QE}	ϵ_{QH}
$\frac{1}{3}$	10	−0.432 841	5	0.074 715	0.085 675	0.030 501
	11	−0.430 623	5	0.075 706	0.084 658	0.030 092
	∞	−0.415 948	—	0.063 177	0.073 724	0.025 813
$\frac{1}{5}$	7	−0.353 494	4	0.016 245	0.020 188	0.009 068
	8	−0.350 066	5	0.015 572	0.019 278	0.008 510
	∞	−0.332 850	—	0.010 516	0.014 912	0.006 288

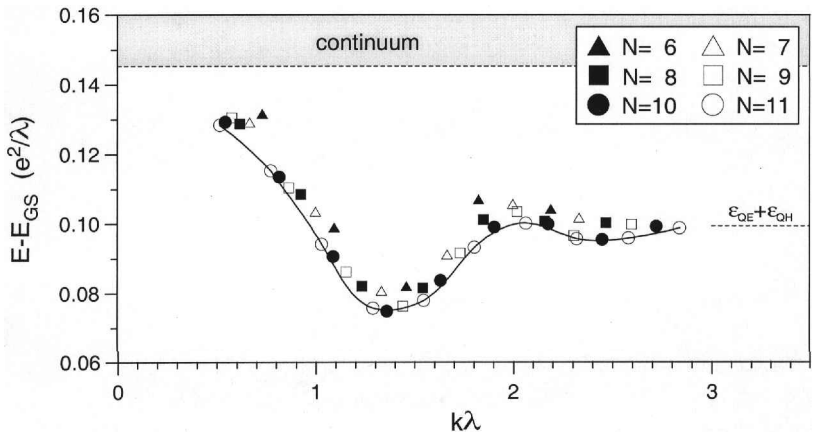

Figure 12. The excitation energy $E - E_{GS}$ as a function of the wave-vector k for the low-lying excitations of the Laughlin $\nu = \frac{1}{3}$ ground state of six to 11 electrons.

except that the probability $\mathcal{G}$ is given as a function of a pair quantum number $\mathcal{R}$ rather than of a pair distance.

In figure 13 (*a*), the $L = 0$ state is the Jain $\nu = \frac{2}{5}$ ground state and the states with $L = 2$, 3 and 4 contain a single QE–QH pair. Similarly, in figure 13 (*b*), the $L = 0$ state is the Laughlin $\nu = \frac{1}{3}$ ground state and the states of a single QE–QH pair have $L = 2$, 3, 4, 5 and 6. Typically for the low-energy states in the lowest LL (or for any other short-range interaction pseudopotential) at $\nu \geqslant \frac{1}{3}$, $\mathcal{G}(1)$ is small and $\mathcal{G}(3)$ is large and, for higher $\mathcal{R}$, $\mathcal{G}$ decreases when $\mathcal{R}$ increases up to the maximum allowed value. The Jain incompressible ground states always have $\mathcal{G}(1)$ smaller than all other states (by at least 0.035 for $N = 6$ and $\nu = \frac{2}{5}$). For Laughlin states, $\mathcal{G}(1)$ is always negligible (less than 0.0008 for $N = 6$ and $\nu = \frac{1}{3}$). The strong maximum of $\mathcal{G}(\mathcal{R})$ at $\mathcal{R} = 3$ means that a large number of pairs are in the '$\nu = \frac{1}{3}$' pair state, on a plane given by the Laughlin correlation factor $(z_1 - z_2)^3$.

In higher LLs, the $\mathcal{G}_{La}(\mathcal{R})$ profiles in figures 13 (*c*)–(*f*) differ from those in the lowest LL, but they are rather similar for different fillings ($\nu = \frac{2}{5}$ and $\frac{1}{3}$). Clearly, at any filling or n, the low-lying states must maximally avoid parentage from pair states of highest repulsion. However, because the pseudopotential $V(\mathcal{R})$ in higher LLs does not increase sufficiently quickly with decreasing $\mathcal{R}$ in its entire range, it appears energetically favourable to minimize total parentage from a number of pair states

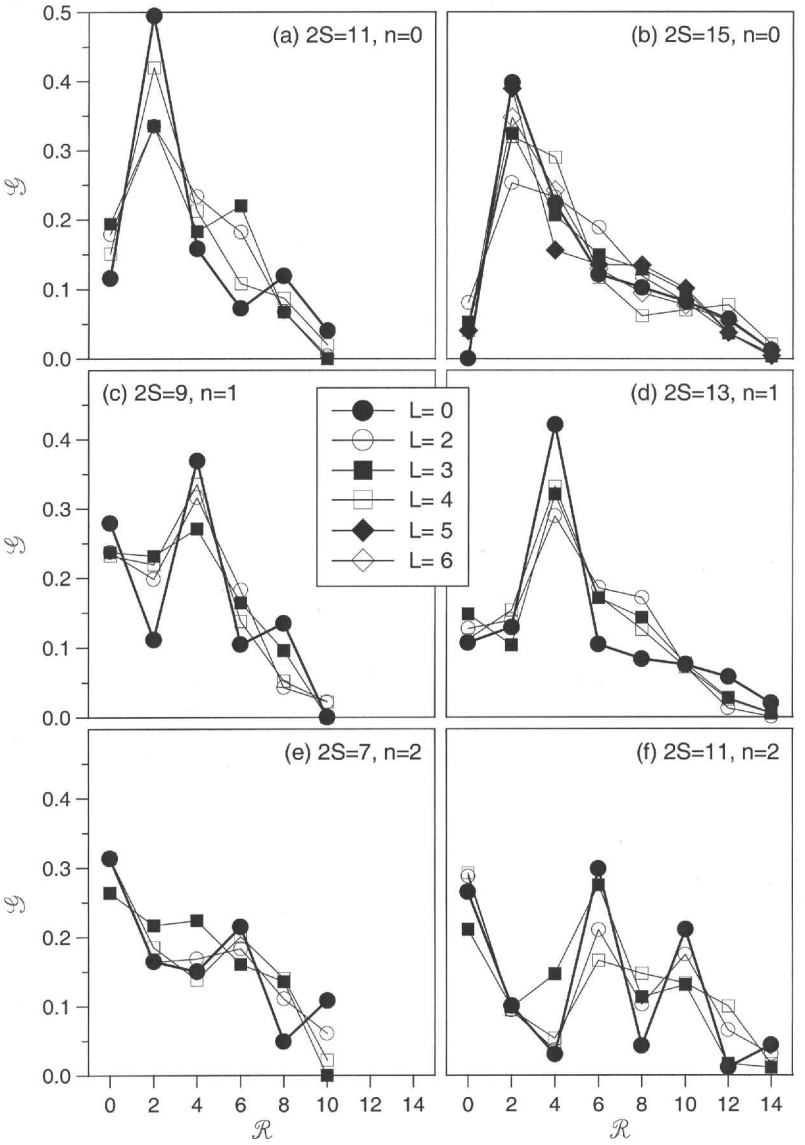

Figure 13. The grandparentage coefficients $\mathcal{G}_{L\alpha}(\mathcal{R})$ as a function of relative pair angular momentum $\mathcal{R}$ for the lowest-energy multiplets of six electrons each with (*a*), (*c*), (*e*) $l = \frac{11}{2}$ and (*b*), (*d*), (*f*) $l = \frac{15}{2}$, calculated for (*a*), (*b*) the lowest (*c*), (*d*) the first excited and (*e*), (*f*) the second excited LLs.

with lowest $\mathcal{R}$, rather from a single highest-energy state with $\mathcal{R} = 1$. It appears that requirement of having small total parentage from a number of pair states of smallest $\mathcal{R}$ (smallest separation) rather than from a single pair state at $\mathcal{R} = 1$ for a density at which only one pair state can be completely avoided implies having large parentage from the $\mathcal{R} = 1$ state. As a result, the maximum of $\mathcal{G}(\mathcal{R})$ shifts from $\mathcal{R} = 3$ (for $n = 0$) to $\mathcal{R} = 5$ (for $n = 1$) or $\mathcal{R} = 7$ (for $n = 2$). Similarly, the minimum at $\mathcal{R} = 1$ (for $n = 0$) shifts to $\mathcal{R} = 3$ (for $n = 1$) or $\mathcal{R} = 5$ (for $n = 2$). The occurrence of a large

number of pairs in certain pair states of small $\mathcal{R}$ (at certain small average distance) and avoiding others defines a different type of short-range correlation in the $\nu = \frac{2}{5}$ or $\nu = \frac{1}{3}$ states in higher LLs. The natural interpretation of the maximum at $\mathcal{R} = 1$ for $n > 0$ instead of the strong minimum as for $n = 0$ seems to be the formation of electron pairs (Haldane and Rezayi 1988, Moore and Read 1991). Since the electron–electron interaction is repulsive, the formation of such pairs is a many body phenomenon and the stability of a pair requires the presence of a surrounding electron gas at an appropriate density. For a given pseudopotential, the pairs could be formed if putting two electrons in a pair state with strong repulsion greatly reduces their interaction with other electron pairs. As a result, the gain in total interaction energy in equation (26) due to reducing the contribution from pair states of intermediate $\mathcal{R}$ can exceed the cost due to creating relatively few (approximately $N/2$) pairs of the smallest $\mathcal{R}$.

At the values of $\mathcal{R}$ at which the pseudopotential $V(\mathcal{R})$ decreases very quickly with increasing $\mathcal{R}$, $V(\mathcal{R})$ is said to have short range. At a given filling factor ν, a number of pair states with largest repulsion are avoided completely, and the dominant contribution to the energy is the largest term in equation (26). This is the one term at the smallest value of $\mathcal{R}$, for which $\mathcal{G}(\mathcal{R})$ does not vanish. There is a strong correlation between energy and the lowest-order non-vanishing CFGP, $\mathcal{G}(2p+1)$ for $(2p+3)^{-1} < \nu \leqslant (2p+1)^{-1}$. The low-energy states always have significantly smaller $\mathcal{G}(2p+1)$ than all other states with $\mathcal{R} \geqslant 2p+1$. As an example, in figure 14 we plot energies and coefficients $\mathcal{G}(1)$ and $\mathcal{G}(3)$ for the eigenstates of six electrons in the lowest LL at $2S = 19$. The band of multiplets indciated as open circles have $\mathcal{G}(1) < 0.0043$, and all other states have $\mathcal{G}(1) > 0.037$. The energy gap between the two bands in figure 14 (*a*) is the result of the CFGP gap in figure 14 (*b*). The states with $\mathcal{G}(1) \approx 0$ are approximate zero-energy eigenstates of the hard-core pseudopotential with $V(1) > 0$ and all other parameters vanishing. In the mean-field CF picture, these states contain four QHs in the Laughlin $\nu = \frac{1}{3}$ state, each with angular momentum $l_{\mathrm{QH}} = \frac{9}{2}$. The angular momentum dependence of energy within this band in figure 14 (*a*) is very similar to that of $\mathcal{G}(3)$ in figure 14 (*c*). In particular, the $L = 0$ ground state, which is the $\nu = \frac{2}{7}$ Jain state in the mean-field CF picture, has the lowest $\mathcal{G}(3)$ of all states in this band.

A closer inspection of figure 14 reveals a general tendency for the energy to increase with increasing L, which does not show up in the $\mathcal{G}(2p+1)$ spectrum. The $\mathcal{G}(2p+1)$ spectrum predicts the relative positions of energy levels with neighbouring L values very well, but, on the average, the energy increases more quickly than $\mathcal{G}(2p+1)$ when L is increased. This is clearly visible in figure 15 (*a*), which shows the energy of six electrons at $l = \frac{11}{2}$ ($\nu = \frac{2}{5}$) and $\frac{15}{2}$ ($\nu = \frac{1}{3}$) as a function of $\mathcal{G}(1)$. States with different angular momenta L are indicated by different symbols, and only five values, $L = 0$, 3, 8, 12, and 18, are shown for clarity. In the lowest LL, the energy and $\mathcal{G}(1)$ and are quite well correlated within each L subspace, and the relations between the two are almost identical for close values of L (e.g. $L = 0$ and 3). However, for very different values of L (e.g., $L = 0$, 8, 12 and 18), the dependence of $\mathcal{G}(1)$ on the energy changes considerably. As found in figure 14, in a pair of states with equal values of $\mathcal{G}(1)$, the state with higher L tends to have a higher energy. Clearly, this is due to the contributions of lower-order terms in equation (26). It will be apparent from equation (38) that the second-highest term, $\mathcal{G}(2p+3)\,V(2p+3)$, increases with L as, roughly, $\mathcal{G}(2p+3) \propto L(L+1)$.

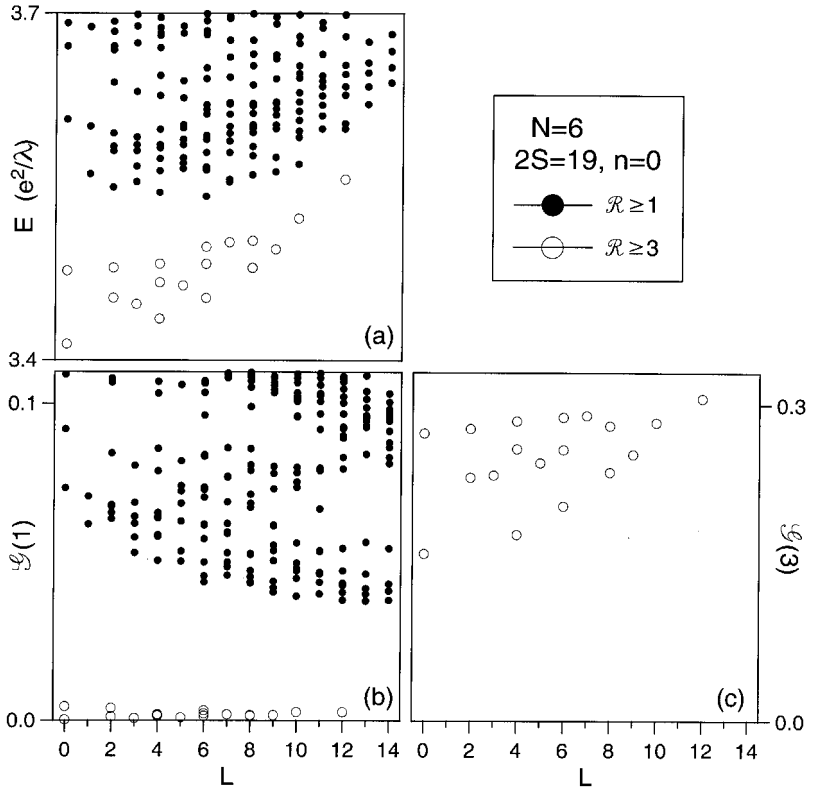

Figure 14. (*a*) The energy E and grandparentage coefficients (*b*) $\mathcal{G}(1)$ and (*c*) $\mathcal{G}(3)$ as functions of the angular momentum L for the system of six electrons in the lowest LL at $2S = 19$: (○), states with $\mathcal{R} \geqslant 3$, that is $\mathcal{G}(1) \approx 0$ and $\mathcal{G}(3) > 0$; (●), states with $\mathcal{R} \geqslant 1$, that is $\mathcal{G}(1) > 0$.

The similarity of the energy and $\mathcal{G}(2p+1)$ spectra makes it clear that a model pseudopotential with only one non-vanishing pseudopotential parameter, $V(1) > 0$, reproduces the main features of the spectrum for $\nu \geqslant \frac{1}{3}$. Similarly, the spectrum of a model pseudopotential with a hard core, $V(1) = \infty$, one finite parameter, $V(3) > 0$, and all higher parameters vanishing resembles the low-energy band of the Coulomb spectrum for $\frac{1}{3} \geqslant \nu \geqslant \frac{1}{5}$. In general, for the filling factor ν in the range $(2p+3)^{-1} < \nu \leqslant (2p+1)^{-1}$, the finite energy eigenstates of the hard-core pseudopotential defined as

$$\begin{aligned} V_{\mathrm{HC}}^{(p)}(\mathcal{R} < 2p+1) &= \infty, \\ V_{\mathrm{HC}}^{(p)}(\mathcal{R} = 2p+1) &= 1, \\ V_{\mathrm{HC}}^{(p)}(\mathcal{R} > 2p+1) &= 0, \end{aligned} \tag{33}$$

are very close to those of the Coulomb pseudopotential. The dependence of finite eigenenergies of $V_{\mathrm{HC}}^{(p)}$ on angular momentum L reproduces the main features of the lowest band of the Coulomb spectrum.

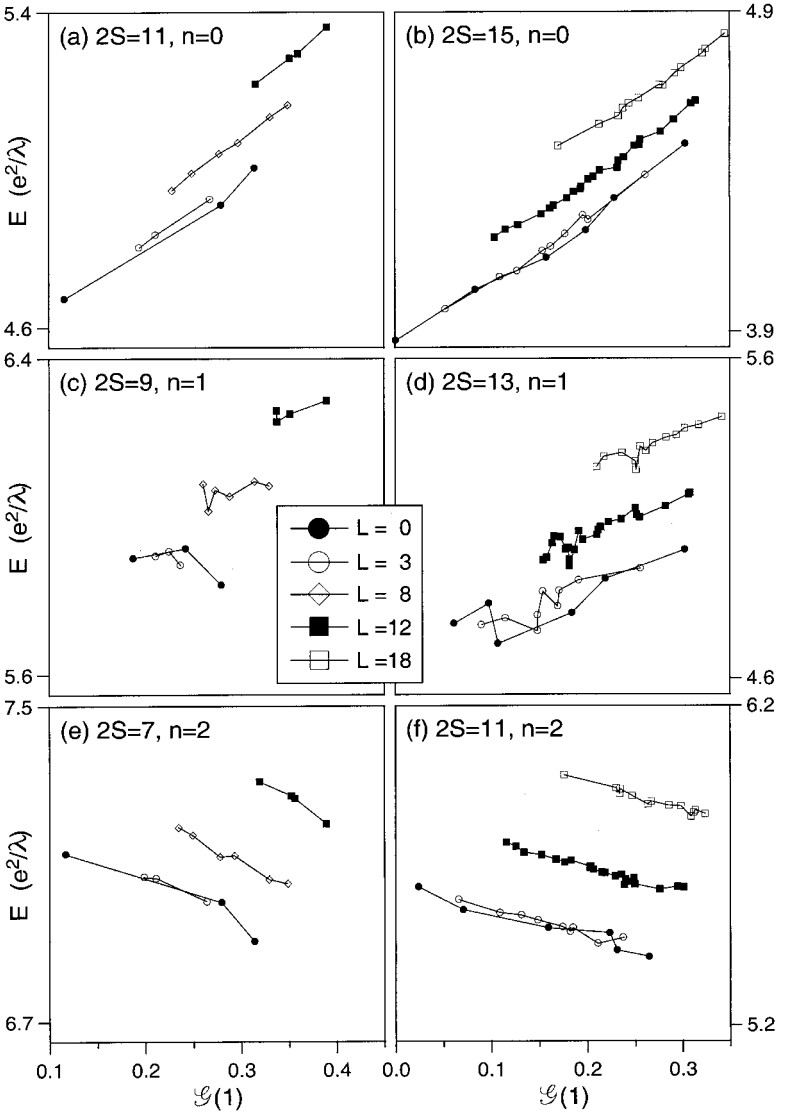

Figure 15. The Coulomb energy of six electrons each with (*a*), (*c*), (*e*) $l = \frac{11}{2}$ and (*b*), (*d*), (*f*) $l = \frac{15}{2}$ as a function of the grandparentage coefficient $\mathcal{G}(1)$, calculated for (*a*), (*b*) the lowest, (*c*), (*d*) the first excited and (*e*), (*f*) the second excited LLs: (●), $L = 0$; (○), $L = 3$; (◇), $L = 8$; (■), $L = 12$; (□), $L = 18$. Only selected values of L are shown.

Because of the different behaviours of the pseudopotential, the above conclusion does not generally hold for higher LLs. The correlation between energy and $\mathcal{G}(1)$ for the same filling factors $\nu = \frac{2}{5}$ and $\frac{1}{3}$ within the first excited LL ($n = 1$), plotted in figures 15 (*c*) and (*d*), is much worse than that for $n = 0$ in figures 15 (*a*) and (*b*). In particular, the lowest-energy $L = 0$ state is no longer the state with the smallest $\mathcal{G}(1)$ at either filling. Also, the Coulomb eigenstates in figures 15 (*c*) and (*d*) are not similar to those of a hard-core repulsion. For example, there is no Laughlin-like state at $\nu = \frac{1}{3}$ with $\mathcal{R} \geqslant 3$ (instead, $\mathcal{G}(1) > 0.06$ for all states) and no Jain-like state at $\nu = \frac{2}{5}$ with $\mathcal{G}(1) \approx 0.12$ (instead, $\mathcal{G}(1) > 0.19$ for all states).

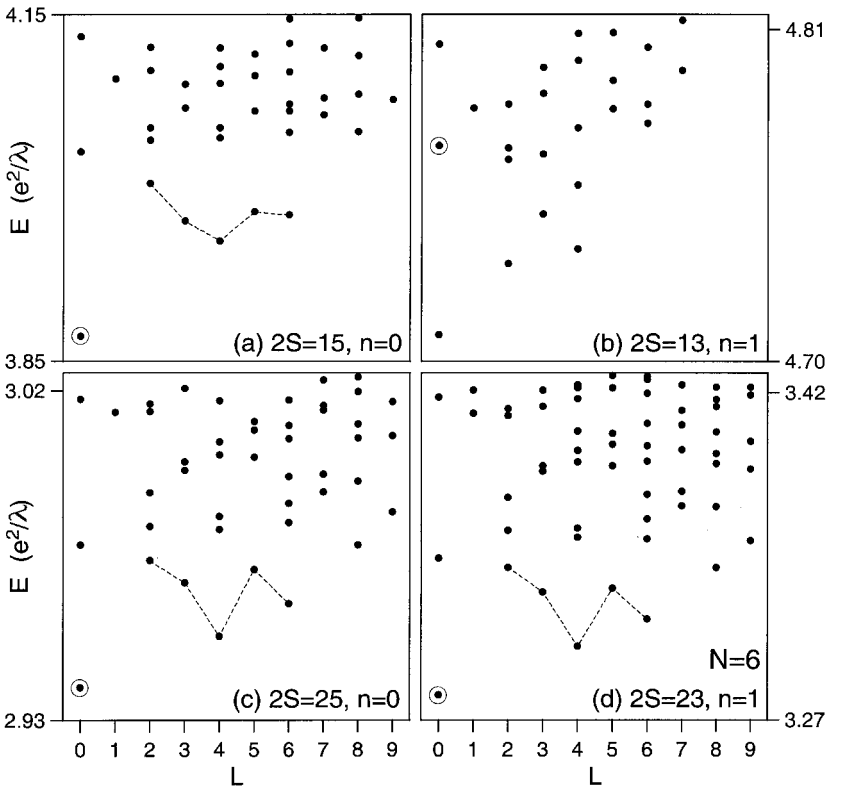

Figure 16. The energy spectra of six electrons each with angular momentum (*a*), (*b*) $l = \frac{15}{2}$ and (*c*), (*d*) $l = \frac{25}{2}$, in (*a*), (*c*) the lowest and (*b*), (*d*) the first excited LLs: (○) states maximally avoiding pairs with largest repulsion; (- - - -), states with one QE–QH pair.

As shown in figures 15 (*e*) and (*f*), the correlation between energy and $\mathcal{G}(1)$ reappears in the second excited LL (Haldane and Rezayi 1985a). However, it is reversed and the low-energy states have high values of $\mathcal{G}(1)$. At $\nu = \frac{2}{5}$, the Jain-like state with $\mathcal{G}(1) \approx 0.12$, maximally avoiding pair states with the smallest average separation and largest repulsion, is the highest-energy state in its $L = 0$ subspace. Similarly, the highest $L = 0$ state at $\nu = \frac{1}{3}$ is the Laughlin-like state with $\mathcal{G}(1) \approx 0.02$.

The approximation of the Coulomb pseudopotential by the hard-core pseudopotential, which gives almost exact many-body eigenstates in the lowest LL and predicts the sequence of the Laughlin incompressible ground states, becomes valid in higher LLs at lower density (filling factor). For $n = 1$ and at fillings $\nu \leqslant \frac{1}{5}$, the second-lowest band ($\mathcal{R} \geqslant 3$) couples to the next higher band ($\mathcal{R} \geqslant 1$). Interband coupling means here that the actual eigenstates are linear combinations of hard-core eigenstates from both bands and the eigenstates originating from the $\mathcal{R} \geqslant 3$ band of the hard-core spectrum have some grandparentage from the $\mathcal{R} = 1$ pair state. However, as seen in figure 8 for only one (ground) state, the band originating from the $\mathcal{R} \geqslant 5$ band is (to a good approximation) uncoupled, that is its eigenstates indeed all have $\mathcal{R} \geqslant 5$ and are very close to the corresponding hard-core states. This occurs because the decoupling of the lowest band from the rest of the spectrum depends on the behaviour of the pseudopotential V at $\mathcal{R} \geqslant 3$, where V for $n = 1$ is similar to that for $n = 0$ (see figure 5). Figure 16 shows the energy spectra of six

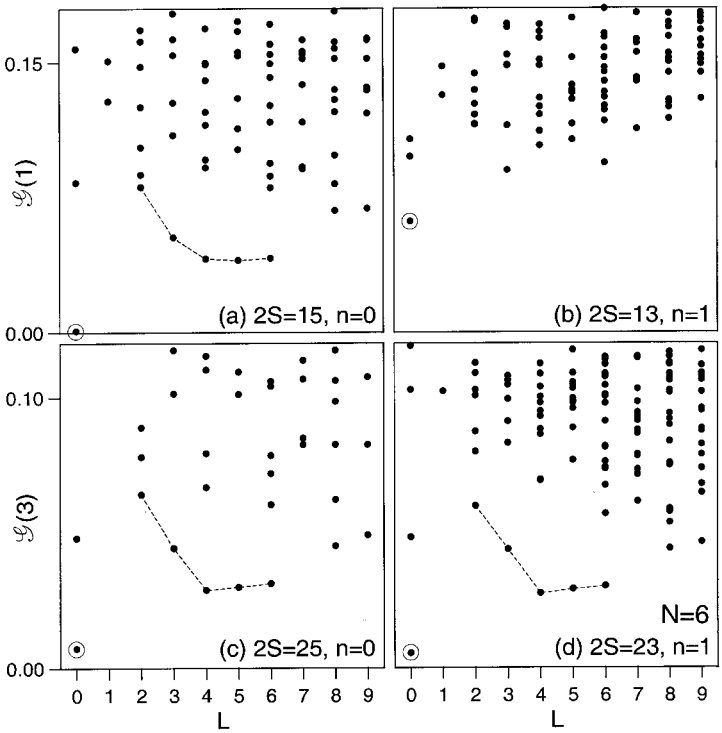

Figure 17. The grandparentage coefficients $\mathcal{G}(1)$ and $\mathcal{G}(3)$ for the eigenstates of six electrons each with angular momentum (*a*), (*b*) $l = \frac{15}{2}$ and (*c*), (*d*) $l = \frac{25}{2}$, in (*a*), (*c*) the lowest and (*b*), (*d*) the first excited LLs: (○), states maximally avoiding pairs with largest repulsion; (- - - -), states with one QE–QH pair.

electrons each with $l = \frac{15}{2}$ (filling factor $\nu = \frac{1}{3}$) and $l = \frac{25}{2}$ ($\nu = \frac{1}{5}$), for the lowest and first excited LL. Figure 17 shows the corresponding spectra of $\mathcal{G}(1)$ and $\mathcal{G}(3)$. States indicated by open circles are states with the lowest $\mathcal{G}(1)$ for $l = \frac{15}{2}$ in figures 16 (*a*) and (*b*) and figures 17 (*a*) and (*b*) and states with $\mathcal{G}(1) \approx 0$ and the lowest $\mathcal{G}(3)$ for $l = \frac{25}{2}$ in figures 16 (*c*) and (*d*) and figures 17 (*c*) and (*d*). Broken lines connect the states that contain a single QP pair in the mean-field CF picture. Clearly, even though the ground states in figures 16 (*a*) and (*b*) and figures 17 (*a*) and (*b*) both have $L = 0$, the two spectra for $l = \frac{15}{2}$ are different. For $n = 1$, the band of states with one QP pair is absent, the ground state is not the state with lowest $\mathcal{G}(1)$, and none of the states has $\mathcal{G}(1) \approx 0$. On the other hand, the two spectra at $l = \frac{25}{2}$ in figures 16 (*c*) and (*d*) and figures 17 (*c*) and (*d*) are very similar. Both contain the band of states with one QP pair, and have the Laughlin $\nu = \frac{1}{5}$ ground states with $\mathcal{G}(1) \approx \mathcal{G}(3) \approx 0$.

§ 7. Relation between the pseudopotential and the occurrence of incompressible ground states

7.1. *Total angular momentum versus average pair angular momentum*

A very useful operator identity

$$\sum_{i<j} \hat{L}_{ij}^2 = \hat{L}^2 + N(N-2)\hat{l}^2 \tag{34}$$

is straightforward to prove (Wójs and Quinn 1999a). Here $\hat{L} = \sum_i \hat{l}_i$ and $\hat{L}_{ij} = \hat{l}_i + \hat{l}_j$. Taking the expectation value of equation (34) in the state $|l^N, L\alpha\rangle$ gives

$$\left\langle \sum_{i<j} \hat{L}_{ij}^2 \right\rangle = L(L+1) + N(N-2)\, l(l+1), \tag{35}$$

which is independent of which multiplet α of a given angular momentum L is being considered. From equation (24) we also have

$$\left\langle \sum_{i<j} \hat{L}_{ij}^2 \right\rangle = \frac{N(N-1)}{2} \sum_{L_{12}} \mathcal{G}_{L\alpha}(L_{12}) L_{12}(L_{12}+1). \tag{36}$$

Combining the above two equations, a non-trivial condition on the allowed values of CFGPs is obtained. Adding the normalization condition following from equation (30), we have the following pair of constraints on the allowed CFGPs profiles $\mathcal{G}_{L\alpha}(\mathcal{R})$ in a multiplet of a given L:

$$\sum_{L_{12}} \mathcal{G}_{L\alpha}(L_{12}) L_{12}(L_{12}+1) = \frac{L(L+1) + N(N-2) l(l+1)}{N(N-1)/2}, \tag{37}$$

$$\sum_{L_{12}} \mathcal{G}_{L\alpha}(L_{12}) = 1. \tag{38}$$

The minimization of the total interaction energy in a Hilbert space of a given N, l, M and L occurs through the optimization of the CFGP profile $\mathcal{G}(\mathcal{R})$ (i.e. the pair correlation function) and must conform to the above constraints.

7.2. *Harmonic repulsive interaction*

It follows from equations (26), (35) and (36) that, if the pseudopotential were given by

$$V_{\mathrm{H}}(L_{12}) = c_1 + c_2 L_{12}(L_{12}+1), \tag{39}$$

all different multiplets with the same value of total angular momentum L would be degenerate at the energy

$$E_{L\alpha} = c_1 N(N-1)/2 + c_2[L(L+1) + N(N-2) l(l+1)]. \tag{40}$$

What is the physical meaning of the pseudopotential V_{H} that is linear in $\hat{L}_{12}^2$? From equation (15), V_{H} is the harmonic interaction:

$$V_{\mathrm{H}}(|\mathbf{r}_i - \mathbf{r}_j|) = c_1' - c_2' \frac{|\mathbf{r}_i - \mathbf{r}_j|^2}{R^2}, \tag{41}$$

Using equation (34), the many-body harmonic interaction Hamiltonian can be written as

$$H_{\mathrm{H}} = \frac{c_1 N(N-1)}{2} + c_2 N(N-2) l(l+1) + B\hat{L}^2, \tag{42}$$

that is, for the harmonic repulsive interaction within an isolated LL, each L subspace is degenerate and the energy increases linearly with increasing $L(L+1)$.

The difference between the harmonic and actual pseudopotentials, the 'anharmonic' contribution $V_{\mathrm{AH}} = V - V_{\mathrm{H}}$, lifts this degeneracy and the actual values of $E_{L\alpha}$ depend on how the values of $\mathcal{G}_{L\alpha}(L_{12})$ are distributed and not just on the

average value of $\hat{L}_{12}^2$ for that value of L. However, if the anharmonic correction V_{AH} is small, the ground state will have the lowest available value of angular momentum, $L = L^{\min}$. If V_{AH} is not small, different multiplets with the same L repel one another, and the splittings caused by V_{AH} can become large when N_L, the number of times that the multiplet L occurs, is large. As a result, the lowest multiplet with certain angular momentum L can have lower energy than multiplets of a smaller neighbouring L', for which $N_{L'} \ll N_L$. In this case, a state with L larger than $L^{\min}$ can become the ground state. For example, for the system of eight electrons at $2S = 22$, the lowest-energy multiplet at $L = 4$ has lower energy than the multiplets at $L = 0$, 1, 2 and 3 (see figure 1 (*b*) and table 6 in § 7.3). Even if V_{AH} is not small, if only $V(L_{12})$ increases with increasing L_{12}, then states with low angular momentum L (and thus low average pair angular momentum L_{12}) will tend to have low energy, and states with high L will tend to have high energy.

How close is the actual Coulomb pseudopotential to the harmonic pseudopotential? The plots of V given as a function of squared pair angular momentum $L(L+1)$ are shown in figure 18. The pseudopotentials for $n = 0$ increase more quickly than linearly with increasing $L(L+1)$ in the entire range of L. For $n = 1$, they do so at low values of L, and the dependence is almost linear close to $\mathcal{R} = 1$.

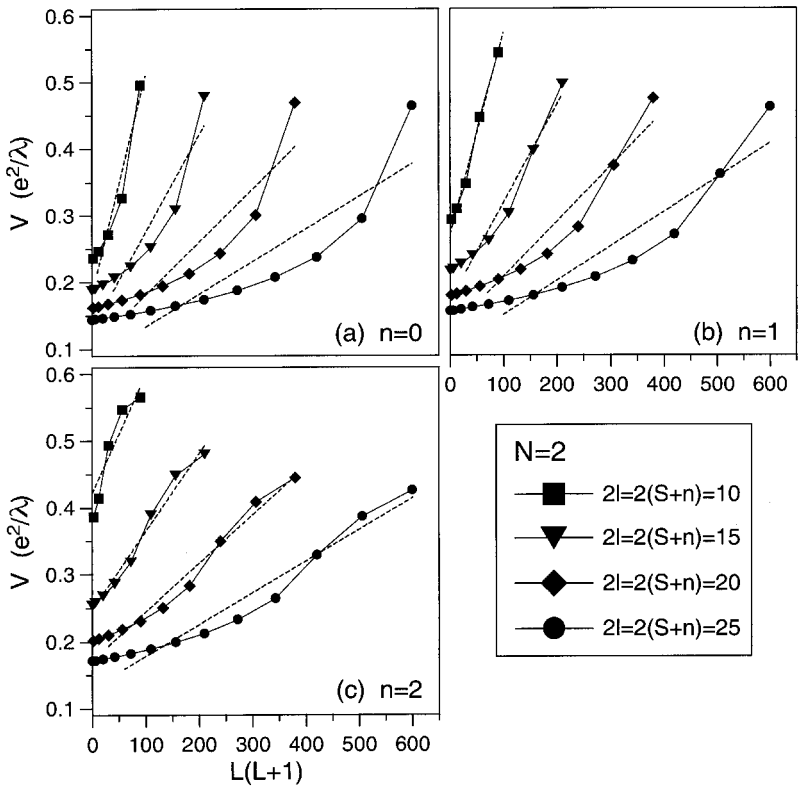

Figure 18. The pseudopotentials of the Coulomb interaction in (*a*) the lowest, (*b*) the first excited and (*c*) the second excited LLs as functions of squared pair angular momentum: (■) $l = 5$; (▼), $l = \frac{15}{2}$; (◆) $l = 10$; (●), $l = \frac{25}{2}$; (- - - -), pseudopotentials corresponding to the the best harmonic interaction fit of the six-electron spectrum, as shown by the broken lines in figure 19 for $l = 5$.

For $n = 2$, V becomes a sublinear function of $L(L+1)$ at high energy. The broken lines give the pseudopotentials of a harmonic interaction which correspond to the best fit to the six-electron spectra.

Examples of energy spectra of the six electron system in the lowest ($n = 0$) and two excited ($n = 1$ and 2) LLs approximated by the harmonic interaction are shown in figure 19 for $l = 5$. The general trend for the energy to increase with L as well as the effects due to level repulsion caused by the anharmonicity of the pseudopotentials are visible. In figure 19, the highest-energy state is the state with the highest L, and the lowest-energy states have low L. The spectrum for $n = 1$ is less distorted from its harmonic fit than the spectra for $n = 0$ and 2. This reflects the fact that the corresponding pseudopotential, indicated by full squares in figure 18 (*b*), is closer to a harmonic pseudopotential than the other two, also indicated by full squares in figures 18 (*a*) and (*c*). For $n = 1$ and 2, the ground state has the lowest available angular momentum $L = L^{\min} = 1$. For $n = 0$, the anharmonicity of the pseudopotential is sufficiently large for the state with $L = 3 > L^{\min}$, to become the ground state due to the level repulsion ($N_3 = 4$ is larger than $N_1 = 2$ or $N_2 = 1$). The open circles in figure 19 indicate the two states at $L = 1$ and 3, which have the lowest

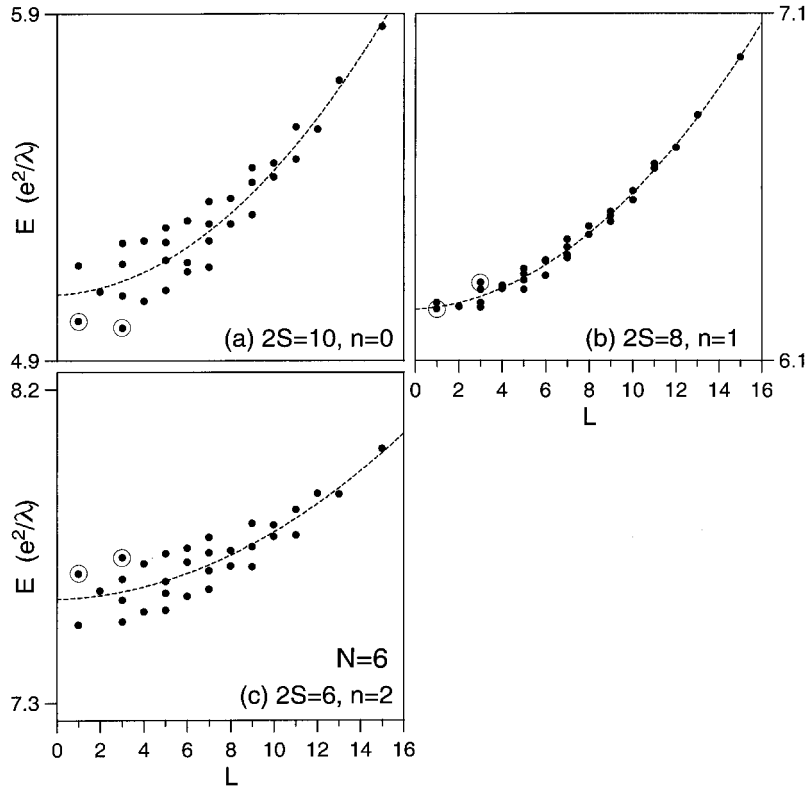

Figure 19. The energy spectra of six electrons, each with $l = 5$, in (*a*) the lowest, (*b*) the first excited and (*c*) the second excited LLs: (●), Coulomb spectra, (- - - -), best harmonic interaction fits to the Coulomb spectra (corresponding harmonic interaction pseudopotentials are given in figure 18); (○) states maximally avoiding pair states with largest repulsion.

$\mathcal{G}(1)$ of all states in the spectrum. For $n = 0$ these states are predicted by the mean-field CF picture as the states of two QEs in the $\nu = \frac{2}{5}$ state.

7.3. *Comparison with the atomic system: Hund's rule*

The problem of electrons in a high magnetic field, occupying single particle states of the nth LL (monopole harmonics with $2S > 0$, shell index $n \ll S$ and angular momentum $l = S + n$), can be compared with that of electrons in an atomic shell, occupying atomic states (spherical harmonics with $S = 0$ and $l = n$). In both cases the problem is that of N electrons each with angular momentum l in a degenerate shell of states with different values of m. However, the pseudopotential $V(\mathcal{R})$ behaves very differently in the two systems. The comparison between the extreme $n = 0$ and $S = 0$ cases is presented in figure 20. The pseudopotentials for the lowest LL shell $V_{n=0}$ and for the atomic shell $V_{S=0}$, calculated for the same $l = S + n$, look quite similar when $V_{S=0}$ is plotted as a function of pair angular momentum L, and $V_{n=0}$ is plotted as a function of the relative pair angular momentum $\mathcal{R} = 2l - L$. Therefore, while $V_{n=0}$ decreases quickly with increasing $\mathcal{R}$ and attains the highest value at $\mathcal{R} = 1$, $V_{S=0}$ does just the opposite.

The pseudopotentials both in figure 20 (*a*) and in figure 20 (*b*) describe the same Coulomb electron–electron interaction $V(r) = e^2/r$, and the origin of this difference lies in the very different Hilbert spaces. The density profiles $\varrho_m(\cos\theta)$ for the single-particle states in both cases are shown in figure 21. θ is the standard spherical coordinate; $z = R\cos\theta$. The average value of z is (Wójs and Quinn 1998a)

$$\langle S,l,m|z|S,l,m\rangle = \frac{mS}{l(l+1)}\,R, \tag{43}$$

$\varrho_{-m}(z) = \varrho_m(-z)$ for the monopole harmonics, and $\varrho_{-m}(z) = \varrho_m(z)$ for the spherical harmonics.

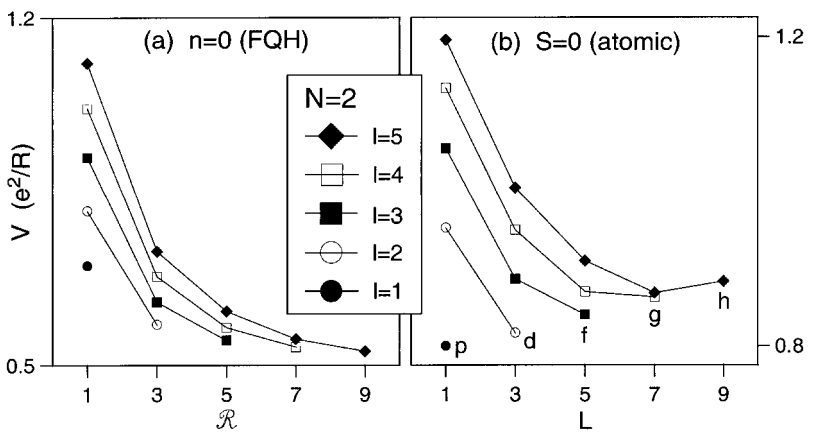

Figure 20. The pseudopotentials V of the Coulomb interaction potential $V(r) = e^2/r$ for a pair of electrons each with angular momentum l: (*a*) lowest Landau level, monopole harmonics, $n = 0$ and $l = S$, V plotted as a function of relative pair angular momentum $\mathcal{R}$; (*b*) atomic shell, spherical harmonics, $S = 0$ and $l = n$, calculated for a radial wavefunction which localizes electrons at radius R, V plotted as a function of pair angular momentum L.

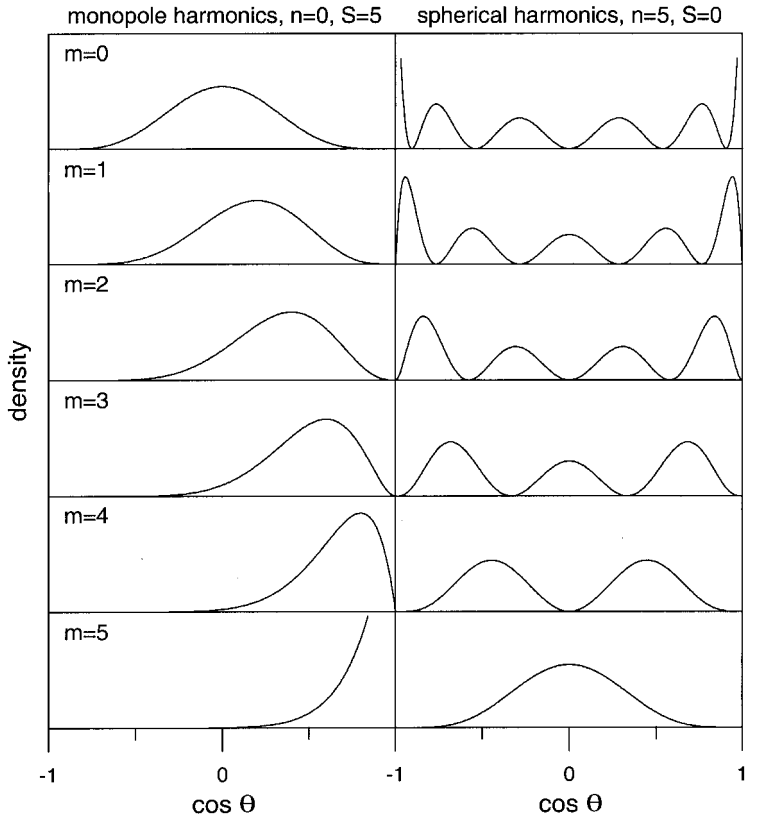

Figure 21. The single-particle density profiles: (*a*) lowest Landau level, monopole harmonics, $n = 0$, $l = S$; (*b*) atomic shell, spherical harmonics, $S = 0$, $l = n$, calculated for a radial wavefunction which localizes the electron at radius R.

The two-electron state $|L,M\rangle$ with the maximum pair angular momentum $L = 2l - 1$ and $M = L$ is the single-particle configuration $|m_1 = l, m_2 = l - 1\rangle$. For the monopole harmonics, it has a high Coulomb energy, as it corresponds to two electrons tightly packed around the north pole of the sphere. On the other hand, the two-electron state with the minimum pair angular momentum, $L = 0$, can be written as $|L = 0, M = 0\rangle = \sum_m \zeta_m |m_1 = m, m_2 = -m\rangle$, that is in each contributing single-particle configuration $|m_1, m_2\rangle$ the two electrons have opposite m values. Opposite m values mean opposite $\langle z\rangle$ values and large spatial separation, and therefore the pair state with $L = 0$ must have a low interaction energy.

For the spherical harmonics, a similar analysis gives opposite answers. The state $|m_1 = l, m_2 = l - 1\rangle$ with the maximum allowed L corresponds to two electrons spread over a large part of the sphere and avoiding one another (a high density for $m = l$ occurs at z corresponding to a low density for $m = l - 1$, and vice versa). Therefore this state must have a low Coulomb energy. In the state with minimum $L = 0$, built of single-particle configurations $|m_1 = m, m_2 = -m\rangle$, opposite m values mean equal density profiles $\varrho(\cos\theta)$, and thus a small average separation and a high interaction energy.

In the case of an atomic system, the reasoning based on equation (34) and the pseudopotential profile leads to Hund's rule. The multiplets with larger total angular

momentum L have, on the average, larger pair angular momenta L_{ij} and thus lower energy. There is only one multiplet with the maximum allowed total angular momentum $L = L^{\max} = Nl - N(N-1)/2$; it is a single-particle 'compact droplet' (maximum density) configuration, for $M = L$ equal to $|m_1, m_2, \ldots, m_N\rangle = |l, l-1, \ldots, l-N+1\rangle$. It has the highest value of the average pair angular momentum and hence it is very likely to be the ground state. A transition to a ground state at a neighbouring lower L would require strong anharmonicity of the pseudopotential. Since relatively low multiplicities N_L occur at L values close to $L^{\max}$ ($N_{L^{\max}-1} = 0$, $N_{L^{\max}-2} \leqslant 1$, $N_{L^{\max}-3} \leqslant 1$, etc.), V_{AH} does not affect the ordering of the levels at high L. Despite this strong indication that the state with the largest L has the lowest energy in atomic systems, Hund's rule is considered an empirical rule, that can be rigorously justified only by detailed numerical calculations. It is also noteworthy that the atomic Hund's rule is usually of interest only for rather low values of l (up to the atomic g or h shell).

By analogy, the opposite rule can be formulated for monopole harmonics (FQH system on a Haldane sphere), stating that the state with the maximum L has the highest energy. Since for monopole harmonics the low-energy states have low values of angular momentum L (with large multiplicities N_L), the direct analogue of the atomic Hund's rule (selecting the ground state) requires that the correction V_{AH} is negligible. Under this assumption it states that the state with lowest available L has the lowest energy. Both rules have been verified numerically.

For the Coulomb interaction acting in the space of monopole harmonics in the lowest LL, the assumption that V_{AH} is negligible does not hold and the multiplicities N_L at low L play a crucial role in determining low-energy L multiplets. In this general case, knowing which multiplet is the ground state or which multiplets form the low-energy sector without performing detailed numerical calculations is a considerably more difficult task. The prescription that the low-energy states are found among states with low values of L which correspond to large N_L can be thought of as a more appropriate analogue to the atomic Hund's rule. As is the case with the atomic Hund's rule, it is an empirical rule that must be verified numerically.

Importantly, the L multiplets for which N_L is relatively large tend to reoccur at the same values of angular momentum L when $2S$ is replaced by $2S^* = 2S - 2p(N-1)$. In table 6 we present, as an example, N_L as a function of L and $2S$ for a system of eight electrons. The values of $2S$ go from zero to 22; the values of L are shown up to eight. The L spaces which are predicted by the CF picture to form the lowest-energy band are underlined. Clearly, they coincide with relatively high multiplicities N_L at the lower values of L. Note, for example, that the high N_L values at $2S = 19$, 20 and 21 appear at the same angular momenta L as the allowed multiplets at $2S^* = 5$, 6 and 7 respectively.

7.4. *Connection between Hund's rule and avoiding pair states of large repulsion*

What is the connection between the two predictions of low-energy states discussed earlier, namely the Hund's rule argument selecting multiplets at low L with high N_L and the argument selecting multiplets that avoid large fractional grandparentage from pair states with largest repulsion? Let us first note that whether a many-body state without grandparentage from certain pair states belongs to the Hilbert space of given N, l, and L depends critically on N_L. It follows from equations (24) and (27) that a multiplet with $\mathcal{G}(\mathcal{R}) = 0$ (e.g. for $\mathcal{R} = 1$) can be constructed if the degeneracy N_L exceeds $N_{\mathcal{R}}$, the number of terms (L', α') in equation (24) with L_{12}

Table 6. The number N_L of independent multiplets at angular momentum L for eight electrons as a function of $2S$ for $0 \leqslant 2S \leqslant 22$. Only values of L up to 8 are included in the table.

	N_L for following L values								
$2S$	0	1	2	3	4	5	6	7	8
0			1						
1	1		1		1				
2	1								
3	1		1		1				
4	1		1	1	1		1		
5	1		1		1		1		
6					1				
7	1								
8					1				
9	1		1		1		1		
10	1		1	1	2	1	2	1	1
11	2		3	1	4	2	4	2	4
12	2	1	4	3	6	5	7	5	7
13	4	1	7	5	11	7	13	9	13
14	4	3	10	9	16	14	19	17	21
15	7	4	16	13	25	21	31	26	35
16	8	8	21	22	35	33	45	42	51
17	12	10	32	30	51	48	66	61	77
18	13	17	42	45	69	70	91	90	108
19	20	22	58	61	96	95	128	124	152
20	22	33	75	85	126	133	169	173	205
21	31	42	101	111	168	175	227	230	277
22	36	59	126	150	215	233	294	307	360

corresponding to $\mathcal{R}$. For example, for $L = 0$, the addition of angular momentum vectors, $\mathbf{L} = \mathbf{L}_{12} + \mathbf{L}'$, selects only one value of L' equal to L_{12}. In this case, it is guaranteed that $N_{\mathcal{R}}$ does not exceed $N'_{L_{12}}$, the number of all $L' = L_{12}$ multiplets of $N-2$ electrons each with angular momentum l. The actual value of $N_{\mathcal{R}}$ can be smaller than $N'_{L_{12}}$ because of the Pauli exclusion principle, which eliminates some of the combinations of $\mathbf{L}'$ and $\mathbf{L}_{12}$. However, $N_{L=0} > N'_{L_{12}}$ guarantees that a multiplet $|l^N, 0\alpha\rangle$, a linear combination of terms in equation (24), can be constructed, for which the coefficients $G_{0\alpha,L_{12}\alpha'}(\mathcal{R})$ vanish simultaneously for all α' and therefore so does the coefficient $\mathcal{G}_{0\alpha}(\mathcal{R})$.

In general, it is difficult to determine $N_{\mathcal{R}}$ by adding dimensions of all relevant L' spaces of $N-2$ electrons because of the Pauli principle which imposes additional constraints on CFGPs in equation (24). However, one can calculate the matrix (α versus $L'\alpha'$) of coefficients $G_{L\alpha,L'\alpha'}(\mathcal{R})$ for all multiplets of given L (for any choice of basis states α, not necessarily the interaction eigenstates) and determine $N_{\mathcal{R}}$ directly. It is clear that N_L must exceed a certain mimimum value for the occurrence of L multiplets which avoid grandparentage from certain (strongly repulsive) pair states. It is also clear that the minimum N_L that is required to exceed $N'_{L_{12}}$ increases with

increasing L since a larger number of angular momenta L' satisfy the addition rule, $|L'-L_{12}| \leqslant L \leqslant L'+L_{12}$, for larger L. If the multiplets with $\mathcal{R} \geqslant 3,5,\ldots$ can be constructed (belonging to the Hilbert space of given N, l and L), they will be the lowest-energy eigenstates of the hard-core interaction defined in equation (17). Hence, the above discussion explains the occurrence of such eigenstates at those of low values of L which have high multiplicity N_L.

Another problem that still needs clarification is whether the multiplets with $\mathcal{R} \geqslant 3,5,\ldots$ are the eigenstates of the actual (not strictly hard-core) interaction pseudopotential $V(\mathcal{R})$ (e.g. the Coulomb interaction in a given LL), and whether they have low energy. In other words, what is the relevant measure of the 'short-range' character of the electron–electron interaction in the lowest LL? Or, what is the condition for $V(\mathcal{R})$ to act like hard-core repulsion and to have the energy spectrum characteristic of the FQH effect, with low-energy states that have $\mathcal{R} \geqslant 3,5,\ldots$? Clearly, whether the ground state and other low-lying multiplets tend to avoid grandparentage from pair states with $\mathcal{R} = 1,3,\ldots$ depends not only on whether $V(\mathcal{R})$ is a decreasing function of $\mathcal{R}$, but also on how quickly it decreases with $\mathcal{R}$. This is because the sequence of CFGPs of a given eigenstate $|L\alpha\rangle$ are mutually connected through the normalization condition given by equations (37), and the non-trivial condition (38). For example, it turns out that the $\nu = \frac{1}{3}$ state with $\mathcal{G}(1) \approx 0$ always has the largest $\mathcal{G}(3)$ of all states. Therefore, $V(\mathcal{R})$ must decrease sufficiently quickly with increasing $\mathcal{R}$ for the state with $\mathcal{R} \geqslant 3$ to be the ground state at the $\nu = \frac{1}{3}$ filling.

7.5. *Definition of the short-range pseudopotential*

The condition for the occurrence of the Laughlin incompressible $\nu = (2p+1)^{-1}$ ground states with $\mathcal{G}(\mathcal{R} < 2p+1) \approx 0$ (and, generally, for the occurrence of low-energy states with $\mathcal{G}(\mathcal{R} < 2p+1) \approx 0$ and low $\mathcal{G}(2p+1)$ for $\nu < (2p+1)^{-1}$) is that the pseudopotential $V(L)$ increases more quickly than linearly with increasing $L(L+1)$. In figures 22 (*c*) and (*d*) we show the energy spectra of a system of six electrons each with angular momentum $l = \frac{15}{2}$, calculated for a model pseudopotential

$$V_\beta(L) = [L(L+1)]^\beta, \tag{44}$$

with $\beta > 1$ and $\beta < 1$. In figures 22 (*a*) and (*b*) we plot the corresponding spectra of the CFGP corresponding to the highest pseudopotential parameter $\mathcal{G}(1)$. The $\mathcal{G}(1)$ spectra look quite similar for $\beta = 1.1$ and 0.9. In particular, in both cases there is one state in the spectrum (indicated by a large open circle) whose $\mathcal{G}(1)$ almost vanishes. At first sight, the energy spectra also look similar. Both of them reveal overall tendency to increase energy with increasing L, and in both of them the larger width of L subspectra coincides with larger N_L. However, a closer inspection shows that the two spectra look like one another's vertical reflections. For $\beta > 1$, the states with low $\mathcal{G}(1)$ tend to have a low energy. For example, within the $L = 0$ subspace, the state with $\mathcal{G}(1) \approx 0$ (large open circle) (this is the Laughlin-like $\nu = \frac{1}{3}$ state) has the lowest energy, and the state with the maximum $\mathcal{G}(1) \approx 0.3$ (large open square) has the highest energy. On the contrary, for $\beta < 1$, the states with low $\mathcal{G}(1)$ tend to have a high energy. For example, for $L = 0$, the state with minimum $\mathcal{G}(1)$ has the highest energy and vice versa. Clearly, the behaviour of the energy as a function of $\mathcal{G}(1)$ is opposite for $\beta > 1$ and $\beta < 1$. This can be demonstrated even more clearly

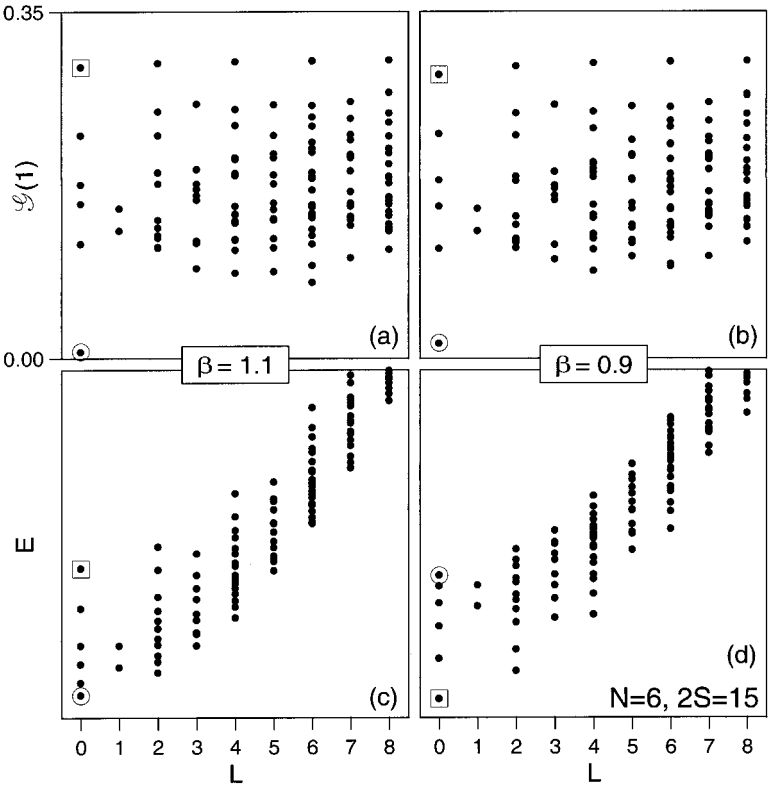

Figure 22. (*a*), (*b*) The grandparentage coefficients $\mathcal{G}(1)$ and (*c*), (*d*) the eigenenergies for a system of six electrons each with angular momentum $l = \frac{15}{2}$, where the interaction pseudopotential is $V_\beta(L) = [L(L+1)]^\beta$, with (*a*), (*c*) $\beta = 1.1$ and (*b*), (*d*) $\beta = 0.9$: (○), $L = 0$ eigenstates with minimum $\mathcal{G}(1)$; (□), $L = 0$ eigenstates with maximum $\mathcal{G}(1)$.

by comparing the expectation values of the V_β interaction in the same states (instead of comparing the eigenspectra). In this case the ordering of energies within each L subspace is exactly reversed.

The exponent β is the relevant measure of the 'short-range' character of a pseudopotential V_β. The condition given by equation (17) that has been used to define an ideal short-range (hard-core) pseudopotential throughout this paper can be rewritten as $\beta \gg 1$. The pseudopotentials with $\beta > 1$ define a class of 'short-range' repulsive interactions characterized by similar behaviours of energy spectra and wavefunctions. For $\beta \to \infty$, the wavefunctions and structure of energy spectra converge to those of the model interaction in equation (17); at the filling factor $\nu = (2p+1)^{-1}$ the ground state is given exactly by the Laughlin wavefunction (or by its spherical form given by Haldane (1983). The pseudopotentials V_β with $\beta < 1$ belong to a separate class of interactions, characterized by their own (common) behaviour of energy spectra and wavefunctions (Wójs and Quinn 1998a), different from those of the short-range class with $\beta > 1$. In particular, Laughlin incompressible $\nu = (2p+1)^{-1}$ ground states with $\mathcal{R} \geqslant 2p+1$ occur only for $\beta > 1$. The harmonic interaction with $\beta = 1$ separates those two classes and does not belong to either.

7.6. *Pseudopotentials of other two-dimensional systems*

The Coulomb pseudopotential for the lowest LL is not strictly of the form $V_\beta(L)$. However, as shown in figure 18 (*a*), it increases more quickly than linearly with an increase in $L(L+1)$ in entire range of L. In consequence, the low-energy states are those with $\mathcal{G}(1) \approx \mathcal{G}(3) \approx \cdots \approx \mathcal{G}(2p-1) \approx 0$ and the lowest value of $\mathcal{G}(2p+1)$, and the $L = 0$ ground states at $2S = (2p+1)(N-1)$ are Laughlin incompressible $\nu = (2p+1)^{-1}$ states. In general, the low-lying states of an interacting many body system at filling factor $\nu \approx (2p+1)^{-1}$ tend to have Laughlin correlations (the states with the lowest energy have vanishing grandparentage from pair states with $\mathcal{R} < 2p-1$ and smallest grandparentage from $\mathcal{R} = 2p-1$), if the pseudopotential $V(\mathcal{R})$ decreases as a function of $\mathcal{R}$ in the entire range and decreases more quickly than the harmonic pseudopotential V_{H} in the vicinity of $\mathcal{R} = 2p+1$. On a sphere, V_{H} increases linearly as a function of the squared pair angular momentum $L(L+1)$; on a plane it decreases linearly as a function of the angular momentum of the relative motion. The condition for Laughlin correlations can be conveniently expressed in terms of the anharmonicity parameter

$$\xi(\mathcal{R}) = V(\mathcal{R}) - V_{\mathrm{H}}(\mathcal{R}), \tag{45}$$

where $V_{\mathrm{H}}(\mathcal{R})$ is the harmonic extrapolation of $V(\mathcal{R}+4)$ and $V(\mathcal{R}+2)$ at $\mathcal{R}$. The condition states that Laughlin correlations (avoiding pairs with $\mathcal{R} \leqslant 2p-1$) occur at $\nu \approx (2p+1)^{-1}$ if $\xi(2p-1) > 0$. In figure 23 we plot $\xi(\mathcal{R})$ for a number of different two-dimensional electron systems in a high magnetic field. By analogy to the electron gas in the lowest LL, one could expect Laughlin-like correlations in these systems, and try to interpret them in terms of mean-field CFs.

As shown in figure 23 (*a*) and in figures 18 (*b*) and (*c*), the electron pseudopotential in excited LLs is of the short range type only for $\mathcal{R} \geqslant 2n+1$. In consequence, the ground states at Laughlin–Jain filling factors $\nu \geqslant (2n+1)^{-1}$ do not have Laughlin correlations (in contrast with the states at the same filling of the lowest LL). Only at lower filling factors, $\nu < (2n+1)^{-1}$, where the part of the pseudopotential which does not decrease quickly enough with increasing $\mathcal{R}$ is completely avoided and does not affect the lowest-energy eigenstates, do these eigenstates have low grandparentage from pair states with large repulsion. In particular, the Laughlin-like incompressible ground states occur only at $\nu < (2n+1)^{-1}$. This explains the compressibility of the $\nu = 2+\frac{1}{3}$, $4+\frac{1}{3}$, and $4+\frac{1}{5}$ ground states (or, at least, different correlations and thus different origins of their incompressibility), and the incompressibility of the $\nu = 2+\frac{1}{5}$ ground state, observed in figure 10.

Another example of a system interacting through the short range pseudopotential is the system of charged excitons (X^-, two electrons bound to a valence hole) or biexcitons (X_2^-, three electrons bound to two valence holes) in the lowest LL, which has been shown to have Laughlin-like incompressible ground states (Wójs *et al.* 1998, 1999a). This is confirmed in figure 23 (*b*), where we also plot $\xi(\mathcal{R})$ calculated for the interaction of an X^- or an X_2^- with an electron (e^-). Note that, for a pair of distinguishable particles, $\mathcal{R}$ can take on any integer value and that the pseudopotentials involving X^- or X_2^- have a hard core ($V = \infty$) at a number of smallest values of $\mathcal{R}$. Clearly, the Laughlin-like e^-–X^- or e^-–X_2^- correlations described by a Jastrow pre-factor in the wavefunction will only occur at odd values of $\mathcal{R}$ (Wójs *et al.* 1999b).

If electrons are confined in parallel two-dimensional layers separated by a small distance d, the interlayer repulsion $V_d(r) = e^2/(r^2+d^2)^{1/2}$ can result in the interlayer

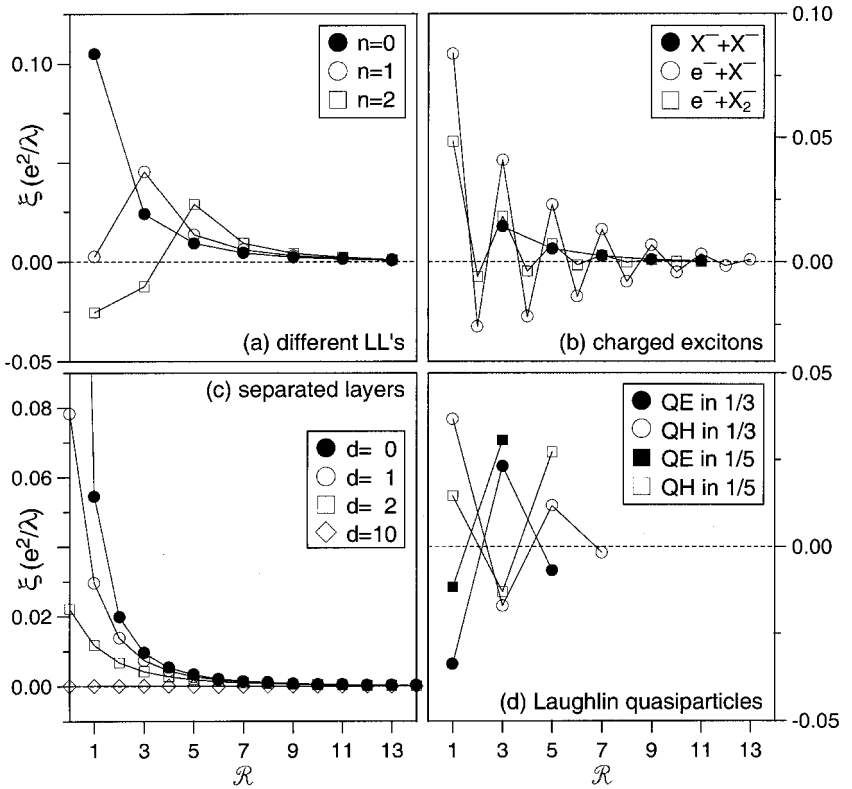

Figure 23. The anharmonicity parameter ξ as a function of relative angular momentum $\mathcal{R}$ for pseudopotentials of different electronic systems in a high magnetic field: (*a*) electrons in different LLs; (*b*) electrons and charged excitons in the lowest LL; (*c*) electrons in two parallel two-dimensional layers separated by d magnetic lengths; (*d*) Laughlin QPs in the $\nu = \frac{1}{3}$ and $\frac{1}{5}$ ground states.

Laughlin correlations, unless d is larger than the characteristic separation between electrons in each layer (approximately $(2\pi/\nu\lambda)^{1/2}$). The plots of $\xi(\mathcal{R})$ for the pseudopotentials $V_d(\mathcal{R})$ in the lowest LL are shown in figure 23 (*c*). When d is large, $V_d(r) \approx [1 - \frac{1}{2}(r/d)^2 + \cdots]/d$ becomes essentially harmonic at small r, $V_d(\mathcal{R})$ becomes harmonic at small $\mathcal{R}$ and the interlayer correlations disappear. Since $V_d(r)$ is a good approximation to an effective two-dimensional potential in a quasi-two-dimensional layer of finite width (about $5d$), figure 23 (*c*) shows also the destruction of the FQH effect in a single wide quantum well (Shayegan *et al.* 1990).

The CF hierarchy uses the mean-field approach for the QPs and therefore should fail when applied to partially filled QP shells unless the QP pseudopotential has a short range. In states with completely filled QE shells (where ν_{QE} is an integer), the gap for creating a new type of QE–QH pair makes the non-degenerate $L = 0$ ground state an incompressible fluid state regardless of the form of the QE pseudopotential. For example, the Jain incompressible $\nu = \frac{2}{5}$ state is obtained when QEs of the $\nu = \frac{1}{3}$ parent state fill one shell ($\nu_{QE} = 1$). For partially filled QP shells, the CF hierarchy correctly predicts daughter incompressible ground states only at certain fractional QP filling factors but not at others. A quick look at the QP pseudopotentials in

figures 11 (*a*) and (*g*) for ten electrons (as well as in figures 1 (*c*) and 2 (*d*) and (*h*) for eight electrons) allows the prediction of filling factors at which the QPs indeed form a Laughlin ground state. In figure 23 (*d*) we plot $\xi(\mathcal{R})$ for QPs of Laughlin $\nu = \frac{1}{3}$ and $\frac{1}{5}$ states, obtained in diagonalization of 11- and eight-electron systems respectively. It can be readily seen that Laughlin QHs should form a stable Laughlin $\nu_{\mathrm{QH}} = \frac{1}{3}$ state of their own. It follows from equation (7) that the $\nu_{\mathrm{QH}} = \frac{1}{3}$ daughter state of the $\nu = \frac{1}{3}$ parent state corresponds to the Jain $\nu = \frac{2}{7}$ state of electrons. Indeed, this state is an incompressible eight-electron ground state in figure 1 (*f*). On the other hand, the $\nu_{\mathrm{QH}} = \frac{1}{5}$ QH state and the corresponding $\nu = \frac{4}{13}$ electron state will be compressible. Indeed, the eight-electron ground state in figure 1 (*d*) does not even have $L = 0$. The $\nu_{\mathrm{QH}} = \frac{1}{7}$ state might be incompressible but with a much smaller gap than that of the $\nu_{\mathrm{QH}} = \frac{1}{3}$, state, which would lead to weak incompressibility of the $\nu = \frac{6}{19}$ electron state. Indeed, the gap above the $L = 0$ ground state of six electrons at $2S = 17$ is very small. For partially filled QE shells, the $\nu_{\mathrm{QE}} = \frac{1}{3}$ ($\nu = \frac{4}{11}$) and $\nu_{\mathrm{QE}} = \frac{1}{7}$ ($\nu = \frac{8}{23}$) states are expected to be compressible, and the $\nu_{\mathrm{QE}} = \frac{1}{5}$ ($\nu = \frac{6}{17}$) state could be weakly incompressible. These predictions are in perfect agreement with numerical results for finite systems (Wójs and Quinn 2000), and we presume that taking into account the behaviour of pseudopotentials of interaction between QEs and between QHs in different stable Laughlin states on all levels of hierarchy explains naturally all the observed odd denominator FQH fillings and allows the prediction of their relative stability without using trial wavefunctions involving multiple LLs and projections on to the lowest LL. The inconsistencies of the original QP hierarchy picture (Haldane 1983, Halperin 1984, Laughlin 1984), namely the appearance of some observed fractions on high hierarchy levels and the actual compressibility of some fractions predicted on lower levels, are removed by noting that Laughlin QPs of a given type form incompressible Laughlin states of their own only at certain filling factors.

7.7. *Prescription for low-energy multiplets*

The discussion presented in the preceding sections can be summarized in the form of a general prescription for the angular momentum multiplets forming the low-energy sector in FQH systems.

(i) The pseudopotential $V(\mathcal{R})$ describing the Coulomb repulsion in an isolated (lowest or excited) LL decreases when the relative pair angular momentum $\mathcal{R}$ increases, that is when the pair angular momentum L_{12} decreases.

(ii) Multiplets with a lower total angular momentum L have a lower expectation value of the pair angular momentum L_{12}, and thus a lower energy.

(iii) The energy levels at the same L repel one another owing to the anharmonicity of $V(\mathcal{R})$. As a result, low values of the total angular momentum L for which many independent multiplets occur are more likely to have some states at lower energy than neighbouring L values with few multiplets.

(iv) Relatively higher multiplicities N_L tend to reoccur at the same values of L for single-particle angular momenta $l^* = l - p(N-1)$. These values coincide with predictions of the mean-field CF picture.

(v) The many-body Hilbert spaces corresponding to low angular momenta L with large multiplicities N_L (as predicted by the mean-field CF picture) contain some states with small grandparentage from pair states of largest repulsion.

(vi) If $V(\mathcal{R})$ decreases more quickly with decreasing $\mathcal{R}$ than the harmonic pseudopotential, the low-lying many-body states avoid grandparentage from pair states of largest repulsion and thus occur at total angular momenta predicted by the mean-field CF picture.

(vii) The gap above the low-energy states that avoid grandparentage from pair states of largest repulsion is governed by the appropriate difference in pseudopotential parameters. This gap does not collapse in the thermodynamic limit.

(viii) At filling factors at which the low-energy band separated from the rest of the spectrum by a gap contains only a non-degenerate (singlet) $L = 0$ ground state, the system is incompressible and exhibits the FQH effect.

§ 8. Summary

We have shown that the success of the mean-field CF picture in correctly and simply selecting the band of lowest-energy multiplets of FQH systems is not due to a cancellation between Coulomb and CS interactions among fluctuations, which are described by totally different energy scales. The reason for the success is related to the nature of the Coulomb pseudopotential $V(\mathcal{R})$ in the lowest LL.

We have identified an exact dynamical symmetry of the hard-core repulsive pseudopotential. Because of this symmetry, the many-body energy spectrum splits into bands of eigenstates which avoid an increasing number of pseudopotential parameters of largest repulsion (the wavefunctions of these eigenstates contain Jastrow pre-factors $\prod_{i<j}(z_i - z_j)^m$ with increasing exponents m). The bands are separated by gaps which are associated with the difference of appropriate pseudopotential parameters and do not collapse in the thermodynamic limit. The incompressibility of Laughlin $\nu = (2p+1)^{-1}$ states in a system with hard-core repulsive interactions results from the fact that the non-degenerate ($L = 0$) ground state is the only state in its (lowest energy) band at these filling factors. The mean-field CF picture can be applied to such systems.

We have defined the class of 'short range' pseudopotentials $V(\mathcal{R})$, for which the Laughlin correlations (avoiding strongly repulsive pair states) minimize the total interaction energy. The occurrence of distinct bands and Laughlin–Jain incompressible ground states in the energy spectrum of systems with short-range interactions is a consequence of weakly broken dynamical symmetry of the hard-core repulsive pseudopotential. The pseudopotential $V(\mathcal{R})$ has a short-range character in a given range of relative pair angular momentum $\mathcal{R}$ if $V(\mathcal{R})$ decreases in this range more quickly as a function of $\mathcal{R}$ than the harmonic pseudopotential. The Coulomb repulsion in the lowest LL belongs to the short-range class in the entire range of $\mathcal{R}$, and hence Laughlin correlations occur at all Laughlin filling factors $\nu = (2p+1)^{-1}$.

We have found that the pseudopotentials in excited LLs decrease more slowly with increasing $\mathcal{R}$ and do not have short-range character at the smallest values of $\mathcal{R}$. As a result, the Laughlin correlations occur in excited LLs only below a certain filling factor. For example, we have shown that the $\nu = 2 + \frac{1}{3}$ state does not have Laughlin correlations in the first excited LL, while $\nu = 4 + \frac{1}{3}$ and $4 + \frac{1}{5}$ states do not have such correlations in the second excited LL. On the other hand, the $\nu = 2 + \frac{1}{5}$ state has Laughlin correlations and an excitation gap comparable with the $\nu = \frac{1}{5}$ state. Because the mean-field CF model describes systems with Laughlin correlations, it is only valid at lower fillings of excited LLs.

The CF hierarchy uses the mean-field approach for the QPs and therefore should fail unless the QP pseudopotential has a short-range nature. We have found that QPs have Laughlin correlations at some of the Laughlin filling factors but not at others. This explains incompressibility of hierarchy ground states at $\nu = \frac{2}{7}$ and compressibility at such hierarchy fractions as $\nu = \frac{4}{11}$ or $\frac{4}{13}$. Also, since the Laughlin QE and QH energies are governed by different electron pseudopotential parameters, the QE energy is higher than the QH energy.

We have also studied the validity of the atomic Hund's rule for systems with different pseudopotentials and shown that a modified Hund's rule remains valid for FQH systems on a Haldane sphere. According to this rule, the FQH states with a small total angular momentum L tend to have a lower energy than states with a large L. This rule is strict for the harmonic interaction for which the energy is completely independent of correlations. Strong anharmonicity of the pseudopotential can invalidate this rule and favour either Laughlin correlated states at low L with large number of multiplets if the pseudopotential has a short-range nature, or other type of correlations (e.g. possible pairing) if the pseudopotential is subharmonic.

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