

Bipartite Composite Fermion States

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We study a class of ansatz wave functions in which composite fermions form two correlated “partitions.” These “bipartite” composite fermion states are demonstrated to be very accurate for electrons in a strong magnetic field interacting via a short-range 3-body interaction potential over a broad range of filling factors. Furthermore, this approach gives accurate approximations for the exact Coulomb ground state at $2 + 3/5$ and $2 + 4/7$ and is thus a promising candidate for the observed fractional quantum Hall states at the hole conjugate fractions at $2 + 2/5$ and $2 + 3/7$.

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While the fractional quantum Hall effect (FQHE) [1] in the lowest Landau level (LL) is securely explained by the composite fermion (CF) theory [2], the physics of the more delicate FQHE in the second LL is currently under debate. The observation [3] of the FQHE at $5/2$ has motivated the idea of pairing of composite fermions, represented by the Pfaffian wave function of Moore and Read [4–7]. Several generalizations of this idea have been proposed [8–11]. We construct below “bipartite” CF (BCF) wave functions at arbitrary fillings by analogy to an earlier theory of CF states in bilayer systems [12] and compare them to exact eigenstates of the Coulomb interaction as well as of a short-range 3-body interaction for which the Pfaffian state is exact at half filling. For the latter, the BCF wave functions are shown to be very accurate over a broad range of filling factors, in particular, for the neutral excitations and quasiparticles of the Pfaffian state, as well as for incompressible FQHE states. The BCF wave functions also provide a good representation of the second-LL Coulomb states at $2 + 3/5$ and $2 + 4/7$. (Evidence has been seen [13] for the FQHE at the particle-hole conjugate states $2 + 2/5$ and $2 + 3/7$.) Aside from the fundamental intrinsic interest in their physical origin, the former state has attracted attention because of a proposal [8] which produces non-Abelian braid statistics for its quasiparticles that is sufficiently complex as to enable, in principle, universal quantum computation.

Our starting point is the observation that, following an identity due to Cauchy, the Pfaffian wave function [4] can be expressed as [5]

$$\Psi_{1/2}^{\text{Pf}} = \mathcal{A} \prod_{j < k}^{N/2} (z_j - z_k)^3 \prod_{j < k}^{N/2} (w_j - w_k)^3 \prod_{j,k=1}^{N/2} (z_j - w_k),$$

where the particles are partitioned into halves, labeled by $z_j = x_j + iy_j$ and $w_k = x_k + iy_k$, and \mathcal{A} denotes the antisymmetrization operator over all N coordinates. (We suppress the ubiquitous Gaussian factor for ease of notation.)

In other words, the Pfaffian wave function is obtained by fully antisymmetrizing the spatial part of Halperin’s 331 bilayer wave function [14]. A more general class of bilayer CF wave functions was constructed by Scarola and Jain [12], and the trial wave functions considered here are constructed by fully antisymmetrizing the spatial part of the generalized bilayer CF wave functions. Explicitly, the BCF wave functions are given by

$$\Psi_{\nu}^{\text{BCF}} = \mathcal{A} \Psi_{\bar{\nu}}^{\text{CF}}(\{z_j\}) \Psi_{\bar{\nu}}^{\text{CF}}(\{w_j\}) \prod_{j,k=1}^{N/2} (w_j - z_k). \quad (1)$$

Prior to antisymmetrization, the wave function has two partitions $\{z_j\}$ and $\{w_j\}$, with different correlations within and across partitions. The factor $\Psi_{\bar{\nu}}^{\text{CF}}(\{z_k\}) = \mathcal{P}_{\text{LLL}} \prod_{j < k}^{N/2} (z_j - z_k)^{2p} \Phi_{\nu^*}$ is Jain’s CF wave function, where Φ_{ν^*} is the wave function of $N/2$ noninteracting electrons at ν^* , \mathcal{P}_{LLL} is the lowest LL (LLL) projection operator, and $\bar{\nu} = \nu^*/(2p\nu^* + 1)$. Composite fermions in different partitions are correlated through the last factor. Power counting tells us that, in the thermodynamic limit, the overall filling fraction ν is related to the CF filling fraction ν^* by

$$\nu = \frac{2\bar{\nu}}{\bar{\nu} + 1} = \frac{2\nu^*}{(2p + 1)\nu^* + 1}. \quad (2)$$

When $\nu^* = n$ is an integer, an incompressible BCF state is obtained at $\nu = 2n/[(2p + 1)n + 1]$. The wave functions for its ground state, neutral excitations, quasiparticles, and quasiholes can be constructed from the corresponding known wave functions of the integral quantum Hall state at $\nu^* = n$. For the special case of $\nu^* = 1$, Eq. (1) reproduces the familiar $1/2p$ Pfaffian ground state.

The BCF wave functions describe complex interactions between composite fermions. Their form suggests pairing correlations, because electrons in the bulk can be added only in pairs (one in each partition), and quasiholes or

quasiparticles can also be created only in pairs. *A posteriori* evidence for the paired nature of Ψ^{BCF} comes from our numerical results below, which demonstrate that they are accurate approximations of the solutions of a 3-body model interaction which has no barrier to forming pairs but a pair repels the approach of a third particle.

Wave functions for incompressible states of the same form as that in Eq. (1) have previously been motivated by Milovanović and Jolicœur [10] and Hermanns [11]. The former considers analogs where the composite fermions in each partition experience a negative flux, and the latter employs a conformal field theory prescription for adding composite fermions in higher Λ levels (i.e., Landau-like levels of composite fermions).

All calculations in this Letter are performed in the standard spherical geometry in which the N electrons move on the surface of the sphere under the influence of a radial magnetic field. The total flux through this spherical surface is $2Qhc/e$, where $2Q$ is an integer due to the Dirac quantization condition. N is taken to be an even integer. The wave functions of Eq. (1) can be translated into the spherical geometry by using standard methods. For Coulomb interaction we consider N electrons in the *second* LL; treating the lowest LL as inert, this system is formally mapped into N electrons in the LLL with an effective interaction. LL mixing and finite thickness corrections may be substantial under experimental conditions [15,16], but we neglect them in the present study. The composite fermions in individual partitions experience an effective flux of $2Q^* = 2Q + 2 - \frac{(2p+1)}{2}N$. The state at $\nu = n/[(2p+1)n \pm 1]$ occurs at $2Q = N/\nu \mp (n+2p)$. The structure of the BCF states is shown schematically in Fig. 1.

The local charge of the quasiparticles, which is the excess charge associated with an isolated quasiparticle, can be determined by asking how many quasiparticles are generated upon the addition of two *electrons*. This produces a local charge of $[(2p+1)n+1]^{-1}$ (in units of the electron charge) for the quasiparticles of the BCF state at $\nu = 2n/[(2p+1)n+1]$. In particular, the local charges of the quasiparticles at $1/2$, $4/7$, and $3/5$ are $1/4$, $1/7$, and $1/10$, respectively.

We have carried out an extensive comparison of the BCF wave functions with the exact eigenstates of a 3-body and the second-LL Coulomb interactions, and we now present the results for the largest systems that we have been able to study. The 3-body interaction [5,6] is given by $\hat{H}_{3\text{-body}} = \sum_{i<j<k} P_{ijk}^{(3)}(3Q-3)$, where $P_{ijk}^{(3)}(L)$ projects the state of the three particles (i , j , and k) into the subspace of total orbital angular momentum L . The BCF wave functions are very complex because of the need for lowest LL projection as well as antisymmetrization, which makes their Monte Carlo evaluation impractical. Fortunately, it is possible to calculate the overlaps and energies of the BCF wave functions exactly if we have the complete set

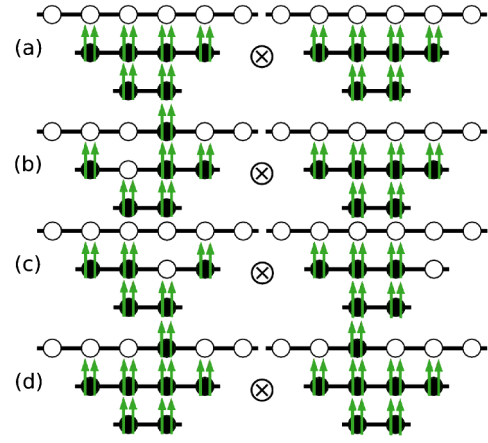


FIG. 1 (color online). Schematic depiction of the incompressible state and excitations at $\frac{4}{7}$ with $2Q^* = 1$. In each panel, the two blocks indicate two partitions, each at filling factor $\nu^* = 2$. Each circle denotes a single particle orbital, and composite fermions occupying them are shown as solid dots with arrows attached to them. The horizontal lines are the Λ levels with the states in each Λ level arranged in the order of increasing L_z . (a) Incompressible state. (b) One neutral excitation. (c) Two quasiholes. (d) Two quasiparticles.

of 3-body or Coulomb eigenstates and eigenenergies. Completeness implies that each BCF state ψ can be expressed as a linear superposition of the exact eigenstates in the appropriate L sector: $\psi = \sum_n c_n |n\rangle$. The coefficients of superposition c_n can be determined by generating a set of linear equations for them by evaluating the wave function for sufficiently many particle configurations $\{z_j\}$. Once expressed explicitly in terms of the interaction eigenstates, the energies and overlaps can be evaluated straightforwardly.

Figure 2 shows the comparison of BCF wave functions for neutral excitations as well as for two and four quasiparticles at $\nu = 1/2$ with the exact eigenstates of the 3-body interaction. Both the energies and overlaps show good agreement for the low energy states, which correspond to states with far-separated (to the extent possible in our finite systems) quasiparticles and quasiholes. (We note that for neutral excitations the separation between the quasiparticle and the quasihole increases with L , whereas for two charged excitations the largest separation is obtained at the smallest L . The situation is more complex when many quasiparticles or quasiholes are present.) Remarkably, the neutral excitation branch is very nicely reproduced beyond a few initial L values. The comparison of the BCF states with the second-LL Coulomb eigenstates at $\nu = 5/2$, also shown in Fig. 2, is less satisfactory. We cannot rule out that the quasiparticles of 3-body and Coulomb interaction are adiabatically connected, although a demonstration of that might require larger system sizes than available here.

The dimension of the Hilbert space spanned by $2n$ quasiparticles or quasiholes of the Pfaffian is of interest.

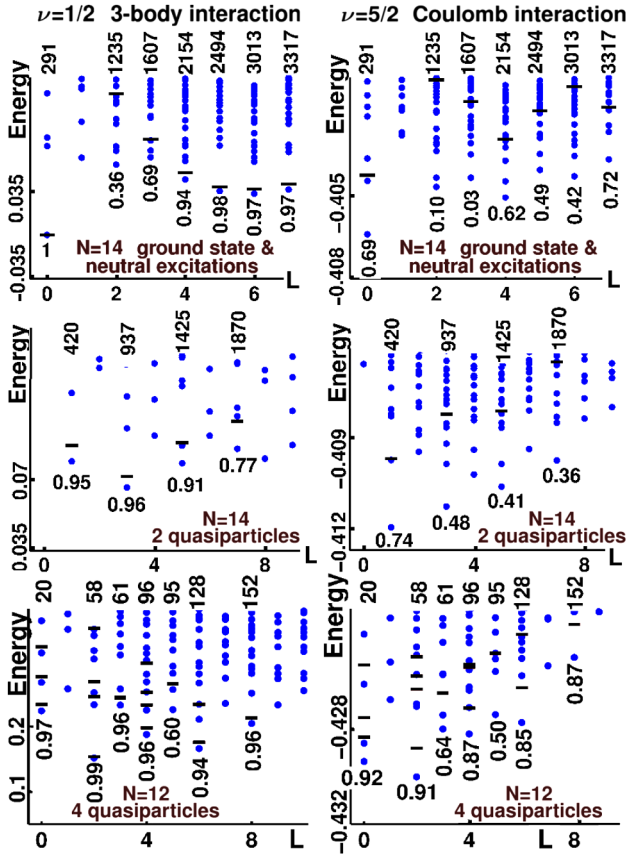


FIG. 2 (color online). Comparison of exact states and the trial functions at $\nu = 1/2$ for the 3-body interaction (left panels) and at $\nu = 5/2$ for the Coulomb interaction (right panels). The blue dots show the exact eigenvalues of the Hamiltonian, and the black lines show the expectation value of energy per particle for the BCF trial wave functions. (When there are several BCF states at a given L , we diagonalize the interaction in that basis.) Here and in the next figure, the number at the bottom indicates the overlap of the lowest energy BCF wave function with the exact lowest energy eigenstate in each L sector, and the integer at the top is the dimension of the L sector. The Coulomb energies per particle are quoted in units of e^2/l , where $l = \sqrt{\hbar c/eB}$ is the magnetic length, and include the interaction with the positively charged background. Top panels: Incompressible state and neutral excitations at $2Q = 2N - 3$; middle: two quasiparticles at $2Q = 2N - 2$; bottom: four quasiparticles at $2Q = 2N - 1$.

Our approach suggests the following counting. For quasiparticles (quasiholes) there are n CFs in the second Λ level (n holes in the lowest Λ level) in each partition. These can be arranged in the $\frac{N-2n}{2} + 2$ single particle orbitals [$(N + 2n)/2$ orbitals] in

$$g_{n\text{-qp}} = \binom{\frac{N-2n}{2} + 2}{n}, \quad g_{n\text{-qh}} = \binom{\frac{N+2n}{2}}{n} \quad (3)$$

distinct ways. Considering both partitions, we get a total of $\frac{1}{2}g(g+1)$ states. On the quasihole side, it is found that these are not all linearly independent, and the dimension of the Hilbert space spanned by them is smaller than the

above number. The space of quasihole states in the BCF formalism can in fact be shown to be identical to the ones studied previously [6,17]. The linear dependences in BCF quasiholes are therefore analogous to those demonstrated by Nayak and Wilczek [17] in the fixed quasihole position basis and have relevance to the braid statistics of the quasiholes. If quasihole states were all independent, that would produce $(2n)!/2(n!)^2$ distinct states (as opposed to the actual 2^{n-1}) for $2n$ quasiholes at fixed locations, as can be obtained, by following Ref. [8], by dividing the full degeneracy by the ‘‘Abelian positional degeneracy’’ $(N/2 + 2n)!/(N/2)!(2n)!$ and taking the thermodynamic limit $N \rightarrow \infty$. For quasiparticles, in contrast, we find that all wave functions constructed above are linearly independent for all $2n$ values that we have tested [18], raising the possibility that quasiparticles have different braiding properties than quasiholes. A possible resolution of this discrepancy is that, although linearly independent, some of the basis states are pushed up to a high energy, and the structure of the low energy subspace is consistent with that of quasiholes. We do not see evidence for the emergence of a low energy band in our numerical results but cannot rule out such a possibility for larger systems.

Other wave functions have been constructed for the quasiparticles of the Pfaffian state. Hansson *et al.* [19] have proposed a wave function that is, in spirit, similar to BCF wave functions. They use a conformal field theory prescription for constructing a CF quasiparticle for $1/3$ [20] and apply it to the Pfaffian wave function in its antisymmetrized bilayer form. Their wave functions are presumably not identical to BCF, however, as indicated by the fact that they obtain the same counting of states for quasiparticles as for quasiholes. Bernevig and Haldane [21] have used certain clustering properties to propose a wave function for quasiparticles of the $5/2$ state. For $\nu = 1/3$, their prescription produces a wave function identical to Jain’s wave function for a single quasiparticle but not for two or more quasiparticles, indicating that the BCF wave functions are in general different from theirs as well.

We next come to the incompressible FQHE states at $\nu = 2n/(3n + 1)$. We consider the fractions $4/7$ and $3/5$, related to states with two and three filled Λ levels, respectively, in each partition, which correspond to total flux $2Q = 7N/4 - 4$ and $2Q = 5N/3 - 5$, respectively. As a first nontrivial test, the exact ground states are uniform, $L = 0$ states at these flux values for all cases we are able to test. Figure 3 displays a comparison of our wave functions with the exact eigenstates for the ground state as well as neutral and charged excitations. For the 3-body interaction, the BCF wave functions have a high overlap with the ground state and the low energy excitations. The results are significant given the fairly large dimensions of the Hilbert space (Fig. 3), demonstrating that the BCF wave functions continue to nicely match the solutions of the 3-body interaction even away from $1/2$.

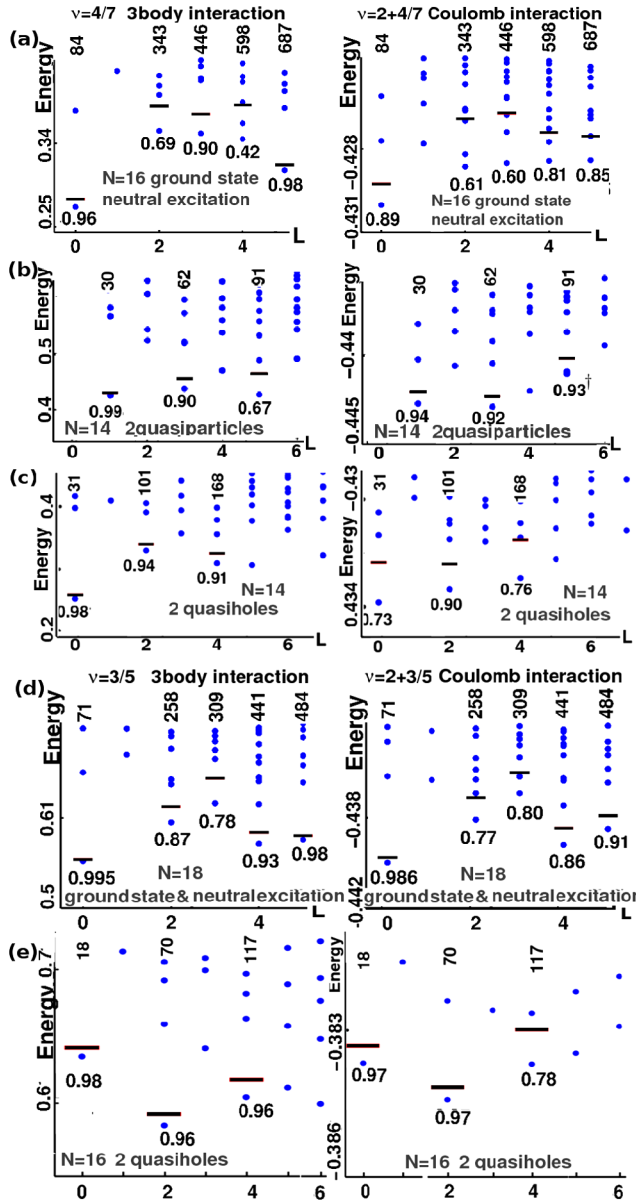


FIG. 3 (color online). Comparison of the BCF wave functions with exact eigenstates of the 3-body interaction (left) and Coulomb interactions (right). The top three rows show the results for (a) the incompressible states and neutral excitations, (b) two quasipoles, and (c) two quasiparticles at $\nu = 4/7$. The bottom two panels show the results for (d) incompressible states and neutral excitations and (e) two quasiholes at $\nu = 3/5$. In the right panel of (b), the overlap at $L = 5$ refers to the projection onto the lowest two almost-degenerate states.

The BCF wave functions also provide a good description of the second-LL Coulomb solutions. Especially notable is the comparison for the $2 + 3/5$ FQHE, where the BCF ground state has an overlap of 98.6% with the exact Coulomb ground state for 18 particles, and its energy (per particle) $-0.43979e^2/l$ deviates by 0.04% from the exact Coulomb energy $-0.43997e^2/l$. The wave function of Read and Rezayi [8], which occurs at $2Q = (5/3)N - 3$,

has overlaps of 0.98 and 0.94 for $N = 15$ and 18 particles, respectively. If we assume particle-hole symmetry, which is exact in the absence of LL mixing, all these results carry over to the hole conjugate state at $2 + 2/5$. Another generalization of the Pfaffian state has been constructed by Bonderson and Slingerland [9] by multiplying $\prod_{j < k} (z_j - z_k)^{2p-1} \mathcal{P}_{LLL} \prod_{j < k} (z_j - z_k)^2 \Phi_n$, the CF wave functions for bosons [22] at $\nu = n/[(2p+1)n \pm 1]$, by the Pfaffian factor. (The $-$ sign refers to negative flux attachment [23].) This produces a $2/5$ state at $2Q = (5/2)N + 2$ which has an overlap of 0.91 with the $N = 14$ ground state [9]. These three states (as well as the standard CF state) occur at different shifts and thus are topologically distinct. Only one of these, if any, may be valid for the actual Coulomb state at $2 + 2/5$, and further investigation, e.g., a comparison of excitations, will be needed to discriminate between them.

Generalization of Eq. (1) to an m th order interpartition zero, which amounts to replacing the cross factor in Eq. (2) by $\prod_{j,k} (z_j - w_k)^m$, will produce BCF states at $2\nu^*/[(2p+m)\nu^* \pm 1]$. Multipartite analogs of Eq. (1) can also be straightforwardly constructed and will represent CF multiplet formation. Turning on the longer range part of the 3-body interaction has been shown to break the pairs to produce free composite fermions [24].

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