

Fractional quantum Hall effect at $\nu = 2 + 4/9$ Ajit C. Balram¹ and A. Wójs²¹*Institute of Mathematical Sciences, HBNI, CIT Campus, Chennai 600113, India*²*Department of Theoretical Physics, Wrocław University of Science and Technology, 50-370 Wrocław, Poland*

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Motivated by two independent experiments revealing a resistance minimum at the Landau level (LL) filling factor $\nu = 2 + 4/9$, characteristic of the fractional quantum Hall effect (FQHE) and suggesting electron condensation into a yet unknown quantum liquid, we propose that this state likely belongs in a parton sequence, put forth recently to understand the emergence of FQHE at $\nu = 2 + 6/13$. While the $\nu = 2 + 4/9$ state proposed here directly follows three simpler parton states, all known to occur in the second LL, it is topologically distinct from the Jain composite fermion (CF) state which occurs at the same $\nu = 4/9$ filling of the lowest LL. We predict experimentally measurable properties of the $4/9$ parton state that can reveal its underlying topological structure and definitively distinguish it from the $4/9$ Jain CF state.

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The fractional quantum Hall effect (FQHE) [1,2] forms a paradigm in our understanding of strongly correlated quantum phases of matter. Of particular interest among the panoply of FQHE phases are the ones observed in the second Landau level (SLL) of ordinary semiconductors such as GaAs. These have attracted widespread attention because of the possibility that the excitations of these phases obey non-Abelian braiding statistics, which could potentially be utilized in carrying out fault-tolerant topological quantum computation [3,4].

FQHE was first observed at filling factor $\nu = 1/3$ [1] in the lowest LL (LLL) and was explained by Laughlin using his eponymous wave function [2]. Soon a whole zoo of fractions were observed, primarily along the sequence $n/(2pn \pm 1)$ (n and p are positive integers) and its particle-hole conjugate [5]. These FQHE states can be understood as arising from the integer quantum Hall effect (IQHE) of composite fermions (CFs) [6,7], which are bound states of electron and an even number ($2p$) of quantized vortices. The theory of weakly interacting CFs captures almost all of the observed FQHE phenomenology in the LLL.

In comparison to the FQHE in the LLL, the FQH states in the SLL are fewer in number and are more fragile [8]. Moreover, the nature of many FQH states in the SLL is dramatically different from their LLL counterparts. In particular, one of the strongest FQH states in the SLL occurs at $\nu = 2 + 1/2$ [8], whereas at the corresponding half-filled LLL a compressible state is observed. A key breakthrough in the field of FQHE came about by a proposal of Moore and Read [9], who posited a “Pfaffian” wave function to describe the state at $\nu = 5/2$. Subsequently, it was understood that the Pfaffian wave function can be interpreted as a p -wave

paired state of composite fermions [10]. The excitations of the Pfaffian state are Majorana fermions which, owing to their non-Abelian braiding properties, could form building blocks of a topological quantum computer [3,4].

The nature of the state at $1/3$ state in the SLL, although believed to be Laughlin-like, has been under intense debate [11–17]. FQHE has been observed at $\nu = 2 + 2/5$ [18–21] but is widely believed to be of a parafermionic nature, unlike the Abelian LLL Jain CF state at $\nu = 2/5$ [22–29]. Furthermore, as yet there is no conclusive experimental evidence of FQHE at the next three members of the $n/(2n + 1)$ Jain sequence, namely $\nu = 3/7, 4/9$, and $5/11$, in the SLL [30], though some features of FQHE have been reported in the literature at some of these fillings [19,20,31]. (A non-Abelian parton state at $3/7$ was constructed in Ref. [32] and shown to be feasible in the SLL.) However, FQHE has been observed at $\nu = 2 + 6/13$ [21]. These observations collectively point to the fact that the nature of FQHE states in the LLL and SLL are different from each other, and a description of FQHE states in the SLL likely entails going beyond the framework of noninteracting composite fermions.

Recently, candidate “parton” states have been constructed to describe the FQHE at many of the experimentally observed fillings in the SLL [33–35]. The parton theory [36] produces model incompressible states beyond the Laughlin [2] and the CF theory [6]. The “ $\bar{n}2111$ ” parton states for $n = 1, 2, 3$ give a good description of the SLL FQHE states observed at $\nu = 2 + 2/3, 2 + 1/2$ and $2 + 6/13$ [33,34]. In particular, the $\bar{n}2111$ sequence underscores the unusual stability of $6/13$ in the SLL.

In this work, we consider the next member of the $\bar{n}2111$ parton sequence, namely $4\bar{2}111$, and consider its feasibility at $\nu = 2 + 4/9$. While FQHE at $2 + 4/9$ has not been established conclusively, indications for it in the form of a minimum in the longitudinal resistance have already been seen in experiments [19,31]. Future experiments on high-quality samples could likely establish FQHE at this filling by an observation of a well-quantized plateau in the Hall resistance. We show that the $4\bar{2}111$ parton state gives a good description

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of the exact SLL Coulomb ground state seen in numerics. Furthermore, the parton state is energetically favorable compared to the 411 Jain CF state at $2 + 4/9$ in the thermodynamic limit. Therefore, if FQHE is established at $\nu = 2 + 4/9$, it is highly likely to be distinct from its LLL counterpart at $\nu = 4/9$, which is well described by the 411 Jain CF state. The parton state is topologically distinct from the Jain CF state and we make predictions for experimentally measurable quantities that can unambiguously distinguish the two. In particular, the parton state supports counterpropagating edge modes that do not occur in the Jain CF state.

The parton theory put forth by Jain [36] constructs FQHE states as a product of IQH states. The essential idea is to break each electron into fictitious objects called partons, place the partons into incompressible IQH states, and recover the final state by fusing the partons back into the physical electrons. The N -electron wave function for the l -parton Jain state, denoted as $n_1 n_2 \dots n_l$, is given by

$$\Psi_{\nu}^{n_1 n_2 \dots n_l} = \mathcal{P}_{\text{LLL}} \prod_{\mu=1}^l \Phi_{n_{\mu}}(\{z_j\}). \quad (1)$$

Here $z_j = x_j - iy_j$ is the two-dimensional coordinate of electron j which is parametrized as a complex number, μ denotes the parton species, and \mathcal{P}_{LLL} implements projection into the LLL. Each parton species is exposed to the external magnetic field and occupies the same area, which fixes the charge of the μ parton species to $q_{\mu} = -e\nu/n_{\mu}$, with $\sum_{\mu} q_{\mu} = -e$, where $-e$ is the charge of the electron. The state $\Phi_{n_{\mu}}$ is the Slater determinant IQH wave function of N electrons filling the lowest $|n_{\mu}|$ LLs. We allow for $n_{\mu} < 0$ and negative values are denoted by \bar{n} , with $\Phi_{\bar{n}} = \Phi_{-|n|} = [\Phi_n]^*$. The parton state of Eq. (1) occurs at a filling factor of $\nu = (\sum_{\mu} n_{\mu}^{-1})^{-1}$ and has a shift [37] of $\mathcal{S} = \sum_{\mu} n_{\mu}$ in the spherical geometry.

The Laughlin state [2] is a “11...” parton state. The parton state $n11$ ($\bar{n}11$) with wave function $\Psi_{n/(2n+1)}^{\text{CF}} = \mathcal{P}_{\text{LLL}} \Phi_n \Phi_1^2$ ($\Psi_{\bar{n}/(2n-1)}^{\text{CF}} = \mathcal{P}_{\text{LLL}} [\Phi_n]^* \Phi_1^2$) correspond to the Jain CF states [6]. Recently, it has been shown that parton states of the form “221...” which are not composite fermion states, could be viable candidates to describe certain FQH states observed in the LLL in wide quantum wells [38] and in LLs of graphene [39,40].

The motivation for considering the $\bar{4}\bar{2}111$ parton state stems from the recent application of parton theory to capture states in the SLL [33–35]. Consider the family of parton states described by the wave function

$$\Psi_{\nu=2n/(5n-2)}^{\bar{n}\bar{2}111} = \mathcal{P}_{\text{LLL}} \Phi_{\bar{n}} \Phi_2 \Phi_1^3 \sim \frac{\Psi_{n/(2n-1)}^{\text{CF}} \Psi_{2/3}^{\text{CF}}}{\Phi_1}. \quad (2)$$

The \sim sign in Eq. (2) indicates that the states written at both sides of the sign differ slightly in the details of the projection. We expect that such details do not change the topological nature of the states [41]. Throughout this text, for the $\bar{n}\bar{2}111$ parton states we use the form given on the rightmost side of Eq. (2). A nice feature of the parton wave functions stated in Eq. (2) is that they can be evaluated for large system sizes which allow a reliable extrapolation of their thermodynamic energies. One can construct the above parton states for large system sizes because the constituent Jain CF states can be

evaluated for hundreds of electrons using the Jain-Kamilla method of projection [7,42–45]. The Jain CF states in this work are evaluated using the Jain-Kamilla method.

The $n = 1$ member, namely $\bar{1}\bar{2}111$, is likely topologically equivalent to the $\nu = 2/3$ $\bar{2}11$ Jain CF state [41]. The $n = 2$ member $\bar{2}\bar{2}111$ has a good overlap with the exact SLL Coulomb ground state at $\nu = 5/2$ [33]. The $n = 3$ state, $\bar{3}\bar{2}111$, gives a good description of the Coulomb ground state at $\nu = 2 + 6/13$ [34]. In the Supplemental Material (SM) [46], we provide further evidence in favor of the feasibility of the $\bar{3}\bar{2}111$ parton state to describe the $\nu = 2 + 6/13$ FQHE. We shall consider the $n = 4$ member of this sequence which occurs at filling factor $4/9$.

Although there is no definitive observation of FQHE at $4/9$ in the SLL, signatures of incompressibility have been seen at $\nu = 2 + 4/9$ and its particle-hole conjugate at $\nu = 2 + 5/9$ [19,31]. FQHE at $\nu = 2 + 4/9$ is likely swamped by a bubble phase [30]; however, it is likely that with improvements in the sample quality or for some interaction parameters close to that of the SLL Coulomb one, FQHE will ultimately be observed at $2 + 4/9$.

For all our calculations, we deploy Haldane’s spherical geometry [47], in which N electrons reside on the surface of a sphere in the presence of a radial magnetic field generated by a monopole of strength $2Q(hc/e)$ located at the center of the sphere. FQHE ground states occur at flux values $2Q = \nu^{-1}N - \mathcal{S}$, where \mathcal{S} is a rational number called the shift, which is useful in characterizing the topological nature of the FQHE state [37]. All FQHE ground states are uniform on the sphere and thus have total orbital angular momentum $L = 0$. The parton states $\Psi_{\nu=2n/(5n-2)}^{\bar{n}\bar{2}111}$ of Eq. (2) satisfy the flux-particle relationship $2Q = [(5n-2)/(2n)]N - (1-n)$; i.e., their filling factors are $\nu = 2n/(5n-2)$ and their shifts are $\mathcal{S} = 1 - n$. Of particular interest to us in this work is the $\bar{4}\bar{2}111$ parton state which has a shift of $\mathcal{S} = -3$. This parton state is topologically distinct from the 411 Jain CF state which also occurs at $\nu = 4/9$ but has a shift of $\mathcal{S} = 6$. We assume a single-component system and neglect the effects of LL mixing and disorder. Under these assumptions, states related by particle-hole conjugation are considered on the same footing.

Throughout this work, we shall write wave functions in the LLL, which is where they are easily evaluated, even though they might apply to states occurring in the SLL. Haldane [47] showed that the physics of the SLL can be simulated in the LLL by using an effective interaction that has the same set of Haldane pseudopotentials in the LLL as the Coulomb interaction has in the SLL. In this work, we have used the form of the effective interaction described in Ref. [48] to simulate the physics of the SLL in the LLL.

Let us begin by testing the viability of the $\bar{4}\bar{2}111$ parton state for $\nu = 2 + 4/9$ FQHE. In Fig. 1 we compare the energies of the $\bar{4}\bar{2}111$ parton and the 411 Jain CF states at $\nu = 4/9$ in the LLL and the SLL. In the LLL, as expected, we find that the Jain CF state has lower energy than the parton state. However, in the SLL we find that the $\bar{4}\bar{2}111$ parton state is energetically more favorable compared to the Jain CF state. For the sake of completeness, we have also investigated the competition between the parton and Jain CF states in the $n = 1$ LL of monolayer graphene. The effective interaction

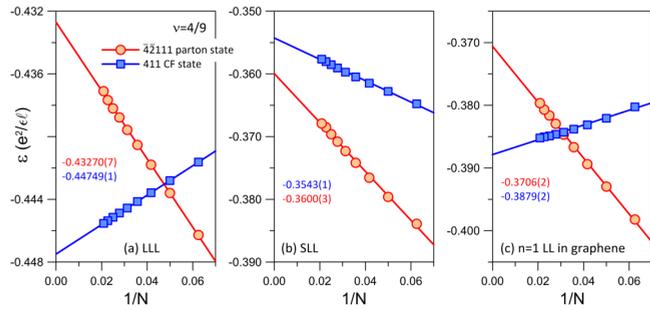


FIG. 1. Thermodynamic extrapolations of the Coulomb per-particle energies for the 411 Jain composite fermion (CF) state and the $\bar{4}2111$ parton state. The left-hand, middle, and right-hand panels show energies for $\nu = 4/9$ in the $n = 0$ LL, $n = 1$ LL of GaAs, and in the $n = 1$ LL of monolayer graphene respectively. The extrapolated energies, obtained from a linear fit in $1/N$, are quoted in units of $e^2/(\epsilon\ell)$ on the plot (error in the fit is indicated in the parentheses). The energies include contributions of electron-background and background-background interactions and are density corrected [52]. The LLL Coulomb energy for the $\nu = 4/9$ Jain CF state has been reproduced from Ref. [53].

we use to simulate the physics of the $n = 1$ LL of monolayer graphene in the LLL is described in Ref. [49]. We find the 411 Jain CF state has lower energy here, consistent with the fact that experimentally observed FQHE states in the $n = 1$ LL of monolayer graphene are well described by the CF paradigm [49–51]. Results for $n = 0$ LL of graphene are identical to those in the LLL of GaAs under our working assumptions of neglecting effects of finite width and LL mixing.

Next, we turn to comparisons of the parton state with the exact SLL Coulomb ground state. The smallest system accessible to exact diagonalization (ED) is that of $N = 16$ electrons at a flux of $2Q = 39$ which has a Hilbert space dimension of 7×10^8 . We have evaluated the ground state for this system with the truncated pseudopotentials from the disk geometry, which differ slightly from the spherical pseudopotentials but are known to provide a more reliable extrapolation to the thermodynamic limit [54,55]. The exact SLL Coulomb ground obtained by using the truncated disk pseudopotentials has $L = 0$. In Fig. 2 we compare the pair-correlation function [56] of this exact SLL Coulomb ground state with that of the $\bar{4}2111$ parton state. Both these pair-correlation functions show oscillations that decay at long distances, which is a typical characteristic of incompressible states [57,58]. Moreover, the two pair-correlation functions are in reasonable agreement with each other. For completeness, we have also evaluated the exact LLL Coulomb ground state for the same system. The overlaps of the LLL and SLL Coulomb ground states obtained using the disk pseudopotentials is 0.3663 and their pair-correlation functions are also very different from each other [46], which indicate that the nature of the ground state in the two LLs are very different.

Currently, we do not have a reliable estimate of the thermodynamic values of the gaps predicted by our parton ansatz. However, we can extract the charge and neutral gaps for $N = 16$ particles from exact diagonalization of the SLL Coulomb interaction at the parton shift. The charge gap here is defined as $[\mathcal{E}(2Q = 40) + \mathcal{E}(2Q = 38) - 2\mathcal{E}(2Q =$

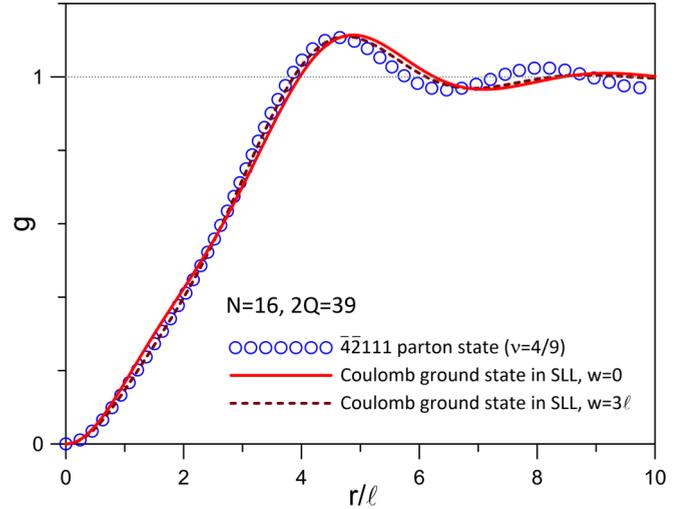


FIG. 2. The pair correlation function $g(r)$ as a function of the arc distance for the exact second Landau level Coulomb ground state obtained using the disk pseudopotentials for width $w = 0$ and $w = 3\ell$, and the $\bar{4}2111$ parton state of Eq. (2) for $N = 16$ electrons at a flux of $2Q = 39$.

$39)]/4$, where $\mathcal{E}(2Q)$ is the exact ground-state energy at flux $2Q$ and the factor of 4 in the denominator accounts for the fact that the addition of a single flux quantum in the parton state produces four fundamental quasiholes. The neutral gap is defined as the difference between the two lowest exact energies at the flux of $2Q = 39$. The charge and neutral gaps for $N = 16$, evaluated using exact diagonalization with the disk pseudopotentials, are $0.009 e^2/\epsilon\ell$ and $0.005 e^2/\epsilon\ell$ respectively, where $\ell = \sqrt{\hbar c}/(eB)$ is the magnetic length and ϵ is the dielectric constant of the background host material.

We next consider the effect of finite width on the system, which we model by taking the transverse wave function to be the ground state for an infinite square quantum well of width w (see SM [46] for details). For the disk pseudopotentials, we find that the ground state for 16 electrons has $L = 0$ for (at least) $w \leq 5\ell$ [46]. Moreover, the pair-correlation function of the exact ground state agrees well with that of the parton state for the entire range of widths considered in this work (see Fig. 2 and Ref. [46]). Furthermore, we find that the system has robust charge and neutral gaps for all the widths considered. We note that the $L = 0$ ground state is delicate. In particular, the exact SLL Coulomb ground state obtained using the spherical pseudopotentials has $L = 2$. However, the overlap between the *lowest energy* $L = 0$ state obtained using the spherical pseudopotentials and the ground state obtained using the disk pseudopotentials is 0.9692, which indicates that these two states are close to each other. Encouragingly, with the spherical pseudopotentials, as the quantum well width is increased the ground state turns uniform in the range $w \in (0.5, 1]\ell$ and stays uniform for $w \in [1, 10]\ell$ [46]. These results indicate that finite thickness enhances the stability of the parton state.

Now that we have made a case for the plausibility of the $\bar{4}2111$ parton state to occur in the SLL, we shall turn to deduce the experimental consequences of this parton ansatz. An additional particle in the factor Φ_4 has charge $e/9$, whereas

that in the factors Φ_2 and Φ_1 has a charge $2e/9$ and $-4e/9$ respectively. All the quasiparticles of the $\bar{4}\bar{2}111$ parton state obey Abelian braid statistics [59]. The $4/9$ Jain CF state is also an Abelian state and hosts quasiparticles of charge $-e/9$ and $-4e/9$.

Next, to infer other topological consequences of the $\bar{4}\bar{2}111$ ansatz, we consider the low-energy effective theory of its edge, which is described by the Lagrangian density [60–63]:

$$\mathcal{L} = -\frac{1}{4\pi} K_{IJ} \epsilon^{\alpha\beta\gamma} a_\alpha^I \partial_\beta a_\gamma^J - \frac{1}{2\pi} \epsilon^{\alpha\beta\gamma} t_I A_\alpha \partial_\beta a_\gamma^I. \quad (3)$$

Here $\epsilon^{\alpha\beta\gamma}$ is the fully antisymmetric Levi-Civita tensor, A is the vector potential corresponding to the external electromagnetic field, a is the internal gauge field, and we have used the Einstein's convention of summing repeated indices. The integer-valued symmetric K matrix and the charge vector t of Eq. (3) for the parton state are given by (see SM [46] for a derivation)

$$K = \begin{pmatrix} -2 & -1 & -1 & 0 & 1 \\ -1 & -2 & -1 & 0 & 1 \\ -1 & -1 & -2 & 0 & 1 \\ 0 & 0 & 0 & -2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad t = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (4)$$

The above K matrix has four negative and one positive eigenvalues and thus the $\bar{4}\bar{2}111$ state hosts four upstream and one downstream edge modes. A naive counting suggests that there are a total of nine edge states for the $\bar{4}\bar{2}111$ ansatz: four from the factor Φ_4 , two from Φ_2 , and one from each factor of Φ_1 . However, these edge states are not all independent since the density variations of the five partons must be identified. This results in four constraints and leads to five edge states consistent with that ascertained from the above K matrix.

Assuming full equilibration of the edge modes, the thermal Hall conductance κ_{xy} at temperatures much smaller than the gap takes a quantized value proportional to the chiral central charge c_- , which is defined as the difference in the number of downstream and upstream modes: $\kappa_{xy} = c_- [\pi^2 k_B^2 / (3h)] T$ [64]. For $\bar{4}\bar{2}111$ ansatz, we thus predict a thermal Hall conductance of $\kappa_{xy} = -3 [\pi^2 k_B^2 / (3h)] T$. The Hall viscosity of the $\bar{4}\bar{2}111$ state is also expected to be quantized [65]: $\eta_H = \hbar \rho_0 \mathcal{S} / 4$, where $\rho_0 = (4/9) / (2\pi \ell^2)$ is the electron density and $\mathcal{S} = -3$ is the shift of the parton state. The ground-state degeneracy of the parton state on a topologically nontrivial manifold with genus g is $|\text{Det}(K)|^g = 18^g$. Besides the $n\bar{n}111$ parton states [17], the $\bar{4}\bar{2}111$ ansatz provides another example of a fully spin polarized Abelian FQH state at $\nu = a/b$ (with

a, b coprime), which has a ground-state degeneracy on the torus that is greater than b .

The $4/9$ Jain CF state is described by the 4×4 K matrix $K = 2C_4 + \mathbb{I}_4$, where C_k is the $k \times k$ matrix of all ones and \mathbb{I}_k is the $k \times k$ identity matrix, and charge vector $t = (1, 1, 1, 1)^T$. In contrast to the $\bar{4}\bar{2}111$ state, assuming the absence of edge reconstruction, the $4/9$ Jain CF state has four downstream edge states and no upstream modes. The $4/9$ Jain CF state thus has a thermal Hall conductance of $\kappa_{xy} = 4 [\pi^2 k_B^2 / (3h)] T$. Moreover, the Hall viscosity of the $4/9$ Jain CF state is given by $\eta_H = (3/2) \hbar \rho_0$, corresponding to shift $\mathcal{S} = 6$. On a manifold of genus g , the $4/9$ Jain CF state has a degeneracy of 9^g .

The presence of upstream neutral modes can be detected in shot noise experiments [66–69]. Recently, thermal Hall measurements have been carried out at several filling factors in the lowest as well as the second LL [70–72]. These experiments can be used to test the predictions of the parton theory and therefore can unambiguously distinguish between the topological nature of the $4/9$ states in the SLL and the LLL. In particular, including the contributions of the filled LLLs of spin up and spin down, the thermal Hall conductance of the $\bar{4}\bar{2}111$ state in the SLL is $-[\pi^2 k_B^2 / (3h)] T$ which is different from what one would expect from the $4/9$ Jain CF state in the SLL, which has $\kappa_{xy} = 6 [\pi^2 k_B^2 / (3h)] T$.

In summary, we have considered the viability of the “ $\bar{4}\bar{2}111$ ” parton state for FQHE at $\nu = 2 + 4/9$ where the first signs of incompressibility in the form of minimum in longitudinal resistance have already been observed experimentally [19,31]. Interestingly, if FQHE eventually stabilizes at this filling factor, then it is likely to be topologically different from its LLL counterpart at $\nu = 4/9$, which is described by a Jain CF state. We also proposed experimental measurements that can reveal the underlying topological structure of the parton state and decisively distinguish it from the $4/9$ state occurring in the lowest Landau level.

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